Adaptive stochastic Galerkin methods for parametric PDEs

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What is this talk about...

- A computational method for uncertainty quantification (stochastic Galerkin FEM)
- Numerical solution of elliptic PDE problems with parametric or uncertain inputs
- Design, analysis, and implementation of adaptive algorithms for computing stochastic Galerkin approximations

Motivation: computational UQ

- Probabilistic description of uncertainty
  - PDEs with random inputs
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  - guaranteed and reliable error control in approximations (rigorous *a posteriori* error analysis)
  - tuning of spatial and stochastic components of approximations
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  - tuning of spatial and stochastic components of approximations
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- Stochastic Galerkin finite element method (SGFEM)
  - *a posteriori* error estimation
  - adaptive algorithms
  - multilevel discretisation
  - convergence and rate optimality

Parametric model problem (1/2)

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$\begin{aligned} -\nabla_x \cdot (a(x, \mathbf{y}) \nabla_x u(x, \mathbf{y})) &= f(x) \qquad && x \in D,\ \mathbf{y} \in \Gamma, \\ u(x, \mathbf{y}) &= 0 && x \in \partial D,\ \mathbf{y} \in \Gamma. \end{aligned} \tag{M}$$

- Domains
  - $D \subset \mathbb{R}^2 \rightsquigarrow$ physical domain
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Remark

Parameters $y_1, y_2, \ldots$ can be seen as images (observations) of independent real-valued random variables with cumulative distribution functions $\pi_1(y_1), \pi_2(y_2), \ldots$. Then, the joint cumulative distribution function is defined as

$$\pi(\mathbf{y}) := \prod_{m=1}^{\infty} \pi_m(y_m), \quad \text{and} \quad \int_{-1}^{1} \mathrm{d}\pi_m(y_m) = \int_{\Gamma} \mathrm{d}\pi(\mathbf{y}) = 1.$$

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  - $0 < a_0^{\min} \le a_0(x) \le a_0^{\max} < \infty \quad$ for almost all $x \in D$
  - $\tau := \frac{1}{a_0^{\min}} \left\| \sum_{m\in\mathbb{N}} |a_m| \right\|_{L^\infty(D)} < 1 \quad \& \quad \sum_{m\in\mathbb{N}} \|a_m\|_{L^\infty(D)} < \infty$

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Remark: $a_0(x)$ typically represents the mean field, i.e., $a_0(x) = \int_\Gamma a(x,\mathbf{y})\mathrm{d}\pi(\mathbf{y})$.

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- Bilinear forms on $\mathbb{V}$
  - $B_0(u, v) := \int_\Gamma \int_D a_0(x) \nabla u(x, \mathbf{y}) \cdot \nabla v(x, \mathbf{y}) \, \mathrm{d}x \, \mathrm{d}\pi(\mathbf{y})$
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Weak formulation: given $f \in L^2(D)$, find $u \in \mathbb{V}$ such that

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[Schwab, Gittelson '11]: the assumptions on $a(x,\mathbf{y})$ ensure the wellposedness of $(\star)$.

Orthogonal polynomials in the parameter domain

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Example

$\nu = (2,5,0,3,0,\dots) \rightsquigarrow \mathrm{supp}(\nu) = \{1,2,4\} \rightsquigarrow P_\nu(\mathbf{y}) = P_2(y_1)\,P_5(y_2)\,P_3(y_4)$

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- Generalized polynomial chaos (gPC) expansion

$$\mathbb{V} \ni v(x,\mathbf{y}) = \sum_{\nu\in\mathcal{J}} v_\nu(x) P_\nu(\mathbf{y}) \quad \text{with unique coefficients} \quad v_\nu \in \mathbb{X}$$

Stochastic Galerkin FEM (1/2)

- Finite dimensional subspace

$$\mathbb{V}_\bullet \subset \mathbb{V} \cong \mathbb{X} \otimes \mathbb{P}$$

Galerkin projection:

$$\text{find} \quad u_\bullet \in \mathbb{V}_\bullet \text{ such that } \quad B(u_\bullet, v_\bullet) = F(v_\bullet) \quad \text{for all } v \in \mathbb{V}_\bullet$$

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- Galerkin orthogonality

$$B(u - u_\bullet, v_\bullet) = 0 \quad \text{for all } v_\bullet \in \mathbb{V}_\bullet$$

- Best approximation property

$$|||\, u - u_\bullet \,||| = \min_{v_\bullet \in \mathbb{V}_\bullet} |||\, u - v_\bullet \,|||, \quad \text{where} \quad |||\, \cdot \,||| := B(\cdot, \cdot)^{1/2}.$$

Stochastic Galerkin FEM (2/2)

- gPC expansion: $u(x, \mathbf{y}) = \sum_{\nu \in \mathcal{J}} u_\nu(x) P_\nu(\mathbf{y})$ with unique coefficients $u_\nu \in \mathbb{X}$
- Discretisation in the parameter domain
  - $\mathcal{P}_\bullet \subset \mathcal{J} \rightsquigarrow$ finite index set
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  - semidiscrete approximation via truncation of gPC expansion

$$u(x, \mathbf{y}) \approx \sum_{\nu \in \mathcal{P}_\bullet} u_\nu(x) P_\nu(\mathbf{y}) \in \mathbb{X} \otimes \mathbb{P}_\bullet \quad \text{with coefficients } u_\nu \in \mathbb{X}$$

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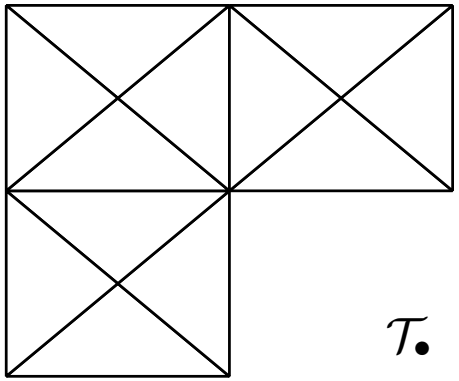
Enhancement of SGFEM approximations (1/2)

- Enhancement of approximations in physical domain
  - initial mesh $\mathcal{T}_0$
  - add new vertices to $\mathcal{T}_\bullet \rightsquigarrow$ mesh refinement
  - mesh refinement by newest vertex bisection (NVB)
  - $\widehat{\mathcal{T}}_\bullet \rightsquigarrow$ uniform refinement of $\mathcal{T}_\bullet$



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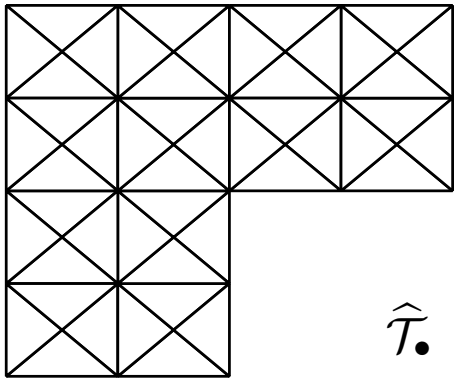

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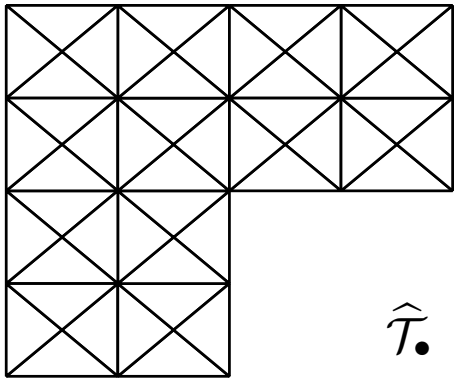

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- Enhancement of approximations in the parameter domain
  - add new indices to $\mathcal{P}_\bullet$
  - finite set $\mathcal{Q}_\bullet \subset \mathcal{J} \setminus \mathcal{P}_\bullet \rightsquigarrow$ detail index set ('boundary' of $\mathcal{P}_\bullet$)
  - $\widehat{\mathcal{P}}_\bullet = \mathcal{P}_\bullet \cup \mathcal{Q}_\bullet \rightsquigarrow$ uniform enrichment of $\mathcal{P}_\bullet$
  - $\widehat{\mathbb{P}}_\bullet = \operatorname{span}\{P_\nu : \nu \in \widehat{\mathcal{P}}_\bullet\} \supset \mathbb{P}_\bullet$

Example

- $\mathcal{P}_\bullet = \{(0,0,\dots);(1,0,\dots);(0,1,0,\dots)\}$

  $\Longrightarrow \mathcal{Q}_\bullet = \{(2,0,\dots);(1,1,0,\dots);(0,2,0,\dots);(0,0,1,0,\dots)\}$

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- $\widehat{\mathbb{V}}_\bullet = (\widehat{\mathbb{X}}_\bullet \otimes \mathbb{P}_\bullet) \oplus (\mathbb{X}_\bullet \otimes \widehat{\mathbb{P}}_\bullet) \supset \mathbb{V}_\bullet$

Auxiliary Galerkin approximation:

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- $|||\, u - u_\bullet \,|||^2 = |||\, u - \widehat{u}_\bullet \,|||^2 + \underbrace{|||\, u_\bullet - \widehat{u}_\bullet \,|||^2}_{\text{error reduction}} \implies |||\, u - \widehat{u}_\bullet \,|||^2 \leq |||\, u - u_\bullet \,|||^2$

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Saturation assumption:

$$\text{there exists } \; 0 < q_{\mathrm{sat}} < 1 \; \text{ such that } \; |||\, u - \widehat{u}_\bullet \,||| \leq q_{\mathrm{sat}} |||\, u - u_\bullet \,|||$$

A posteriori error estimation in SGFEM

- Estimator structure

$$\eta_\bullet^2 \approx |||\, u - u_\bullet \,|||^2 = (\mathbb{X}\text{-error})^2 + (\mathbb{P}\text{-error})^2$$

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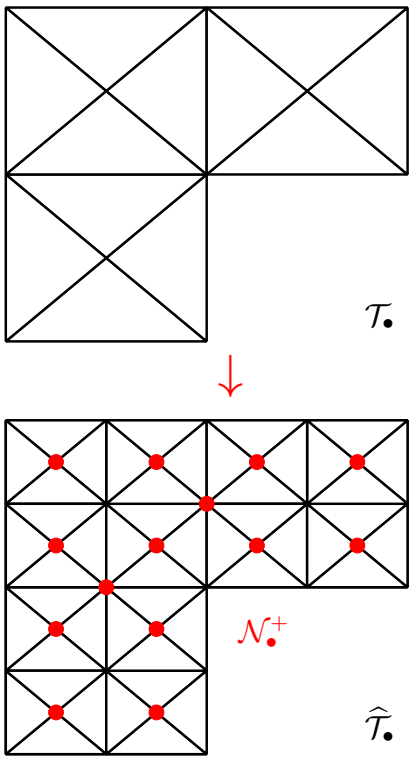
- *A posteriori* error estimation for SGFEM
  - residual-based error estimators
    [Eigel, Gittelson, Schwab, Zander '14]
  - hierarchical error estimators
    [B., Powell, Silvester '14], [B., Silvester '16]
  - two-level error estimators
    [B., Praetorius, Rocchi, Ruggeri; CMAME '19]

A posteriori error estimation: spatial estimator

- Two-level spatial error estimator
  [Mund, Stephan, Weiße '98], [Mund, Stephan '99]
  - $\mathcal{N}_\bullet^+ \rightsquigarrow$ set of interior midpoints
  - $\widehat{\varphi}_{\bullet,z} \in \widehat{\mathbb{X}}_\bullet \rightsquigarrow$ hat function associated with $z \in \mathcal{N}_\bullet^+$
  - $\eta_\bullet^2(z) = \sum_{\nu \in \mathcal{P}_\bullet} \frac{|F(\widehat{\varphi}_{\bullet,z} P_\nu) - B(u_\bullet, \widehat{\varphi}_{\bullet,z} P_\nu)|^2}{\|a_0^{1/2} \nabla \widehat{\varphi}_{\bullet,z}\|^2_{L^2(D)}}$
  - $(\mathbb{X}\text{-error})^2 \approx \sum_{z \in \mathcal{N}_\bullet^+} \eta_\bullet^2(z)$



A posteriori error estimation: parametric estimator

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  [B., Silvester '16]
  - $\nu \in \mathcal{Q}_\bullet$ (N.B.: $\mathcal{Q}_\bullet$ is the boundary of the index set $\mathcal{P}_\bullet$)

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  - $\eta_\bullet^2(\nu) = \|e_{\bullet\nu}\|_{B_0}^2 := B_0(e_{\bullet\nu}, e_{\bullet\nu})$
  - $(\mathbb{P}\text{-error})^2 \approx \sum_{\nu \in \mathcal{Q}_\bullet} \eta_\bullet^2(\nu)$

A posteriori error estimation: main results

[B., Praetorius, Rocchi, Ruggeri; CMAME '19]

- $\eta_\bullet^2 = (\text{estimated } \mathbb{X}\text{-error})^2 + (\text{estimated } \mathbb{P}\text{-error})^2 = \sum_{z \in \mathcal{N}_\bullet^+} \eta_\bullet^2(z) + \sum_{\nu \in \mathcal{Q}_\bullet} \eta_\bullet^2(\nu)$

Theorem 1 (equivalence of total error estimate and error reduction)

There exists $C = C(a_0, \tau, \mathcal{T}_0) \geq 1$ such that

$$C^{-1}\, \eta_\bullet^2 \leq |||\, \widehat{u}_\bullet - u_\bullet \,|||^2 \leq C\, \eta_\bullet^2$$

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Corollary (efficiency & reliability)

- $|||\, u - u_\bullet \,|||^2 \geq C^{-1}\, \eta_\bullet^2$ (efficiency)
- saturation assumption $\Longrightarrow$ $|||\, u - u_\bullet \,|||^2 \leq \dfrac{C}{1 - q_{\mathrm{sat}}^2}\, \eta_\bullet^2$ (reliability)

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Remark

- $\eta_\bullet(z)$ is associated with an interior edge midpoint $z \in \mathcal{N}_\bullet^+$
- $\eta_\bullet(\nu)$ is associated with a new index $\nu \in \mathcal{Q}_\bullet$

These are local *error reduction indicators* for
spatial refinement / parametric enrichment $\implies$ key to adaptivity

Adaptive SGFEM algorithm

Adaptive algorithm

`INPUT:` initial mesh $\mathcal{T}_0$ & initial index set $\mathcal{P}_0 = \{(0, 0, \dots)\}$

`FOR` $\ell = 0, 1, 2, 3, \dots$ `DO:`

- `SOLVE:` compute approximation $u_\ell \in \mathbb{V}_\ell$ for mesh $\mathcal{T}_\ell$ and index set $\mathcal{P}_\ell$
- `ESTIMATE:` compute *local* error indicators and the *total* error estimate
  - spatial & parametric indicators $\rightsquigarrow$ $\{\eta_\ell(z);\ z \in \mathcal{N}_\ell^+\}$, $\{\eta_\ell(\nu);\ \nu \in \mathcal{Q}_\ell\}$
  - energy error estimate $\eta_\ell = \sum_z \eta_\ell^2(z) + \sum_\nu \eta_\ell^2(\nu)$
- `MARK:` mark certain vertices $\mathcal{M}_\ell \subseteq \mathcal{N}_\ell^+$ and indices $\mathcal{R}_\ell \subseteq \mathcal{Q}_\ell$
- `REFINE:` enhance approximation space
  - mesh refinement (NVB) $\rightsquigarrow$ $\mathcal{T}_{\ell+1} = \mathsf{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$
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`OUTPUT:` stochastic Galerkin approximations $\{u_\ell\}$ and error estimates $\{\eta_\ell\}$

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  - `IF` $\eta_\ell <$ `tol THEN STOP`
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Dörfler marking strategy

- Dörfler marking for a generic sequence [Dörfler '96]
  - Let $\{\beta_s;\ s = 1, 2, \ldots, S\}$ be a generic sequence (e.g., error indicators).
  - Given $\theta \in (0, 1]$, build a subset $\mathcal{M} \subseteq \{1, 2, \ldots, S\}$ of minimal cardinality such that $\{\beta_s;\ s \in \mathcal{M}\}$ is the set of $\#\mathcal{M}$ largest elements of the original sequence and

$$\sum_{s \in \mathcal{M}} \beta_s^2 \geq \theta \sum_{s=1}^{S} \beta_s^2.$$

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- MARK: mark certain vertices $\mathcal{M}_\ell \subseteq \mathcal{N}_\ell^+$ and indices $\mathcal{R}_\ell \subseteq \mathcal{Q}_\ell$
  - Dörfler marking for $\left\{\eta_\ell(z);\ z \in \mathcal{N}_\ell^+\right\}$ with $\theta_{\mathbb{X}} \in (0, 1]$ yields $\mathcal{M}_\ell \subseteq \mathcal{N}_\ell^+$
  - Dörfler marking for $\left\{\eta_\ell(\nu);\ \nu \in \mathcal{Q}_\ell\right\}$ with $\theta_{\mathbb{P}} \in (0, 1]$ yields $\mathcal{R}_\ell \subseteq \mathcal{Q}_\ell$

Refinement criterion

Energy error estimate: $\quad |||\, u - u_\ell \,|||^2 \simeq \eta_\ell^2 = \sum_{z \in \mathcal{N}_\ell^+} \eta_\ell^2(z) + \sum_{\nu \in \mathcal{Q}_\ell} \eta_\ell^2(\nu).$

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$$|||\, \widetilde{u}_\ell^{\mathbb{X}} - u_\ell \,|||^2 \simeq \sum_{z \in \mathcal{M}_\ell} \eta_\ell^2(z) \quad \text{and} \quad |||\, \widetilde{u}_\ell^{\mathbb{P}} - u_\ell \,|||^2 \simeq \sum_{\nu \in \mathcal{R}_\ell} \eta_\ell^2(\nu),$$

where $\widetilde{u}_\ell^{\mathbb{X}}$ and $\widetilde{u}_\ell^{\mathbb{P}}$ are two *enhanced* Galerkin approximations.

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    perform mesh refinement (NVB) $\rightsquigarrow \mathcal{T}_{\ell+1} = \mathsf{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$
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    perform parametric enrichment $\rightsquigarrow \mathcal{P}_{\ell+1} = \mathcal{P}_\ell \cup \mathcal{R}_\ell$

Convergence of adaptive SGFEM

Energy error estimate: $|||\, u - u_\ell \,|||^2 \simeq \eta_\ell^2 = \sum_{z \in \mathcal{N}_\ell^+} \eta_\ell^2(z) + \sum_{\nu \in \mathcal{Q}_\ell} \eta_\ell^2(\nu).$

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Theorem 2 (convergence of error estimates)

For any $\theta_{\mathbb{X}}, \theta_{\mathbb{P}} \in (0, 1]$ $\implies$ $\lim_{\ell \to \infty} \eta_\ell = 0.$

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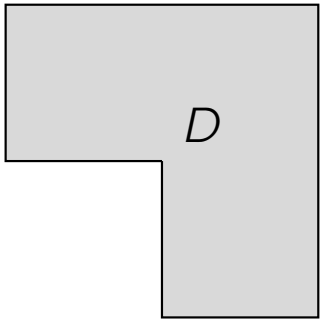
Remarks

- Theorem 2 provides a theoretical guarantee that, for any given positive tolerance, the adaptive algorithm stops after a finite number of iterations.
- No assumptions on the Galerkin approximations generated by the algorithm.
- The result extends to more general marking strategies.

Numerical results: test problem set-up



- $-\nabla \cdot (a \nabla u) = 1$ in $D \times \Gamma$, $\quad u = 0$ on $\partial D \times \Gamma$
- $D := (-1,1)^2 \setminus (-1,0]^2 \rightsquigarrow$ L-shaped domain
- Diffusion coefficient [Eigel, Gittelson, Schwab, Zander '14]:

$$a(x, \mathbf{y}) = 1 + \sum_{m=1}^{\infty} y_m \left(A m^{-2} \cos(2\pi \beta_1(m) x_1) \cos(2\pi \beta_2(m) x_2) \right),$$

$$A = 0.547, \quad \beta_1(m) + \beta_2(m) =: k_m \in \{1,1;\ 2,2,2;\ 3,3,3,3;\ 4,4,4,4,4;\ \ldots\}$$

- $\{y_m\}_{m \in \mathbb{N}}$ are images of $U(-1,1)$ iid mean-zero r.v. $\implies \mathrm{d}\pi_m(y_m) = \frac{1}{2}\,\mathrm{d}y_m$.

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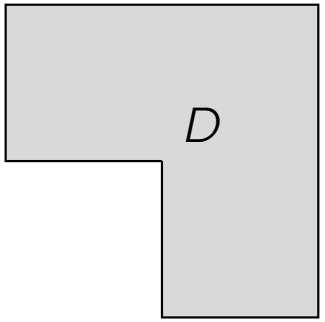

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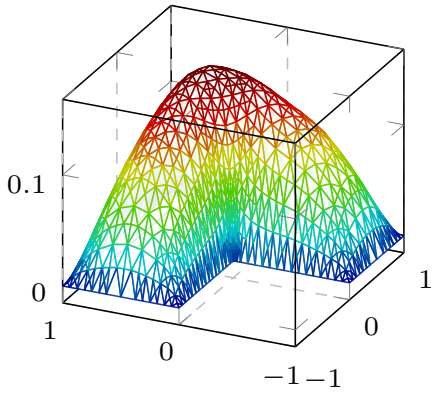
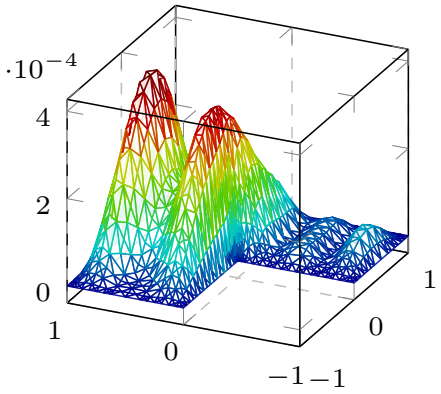
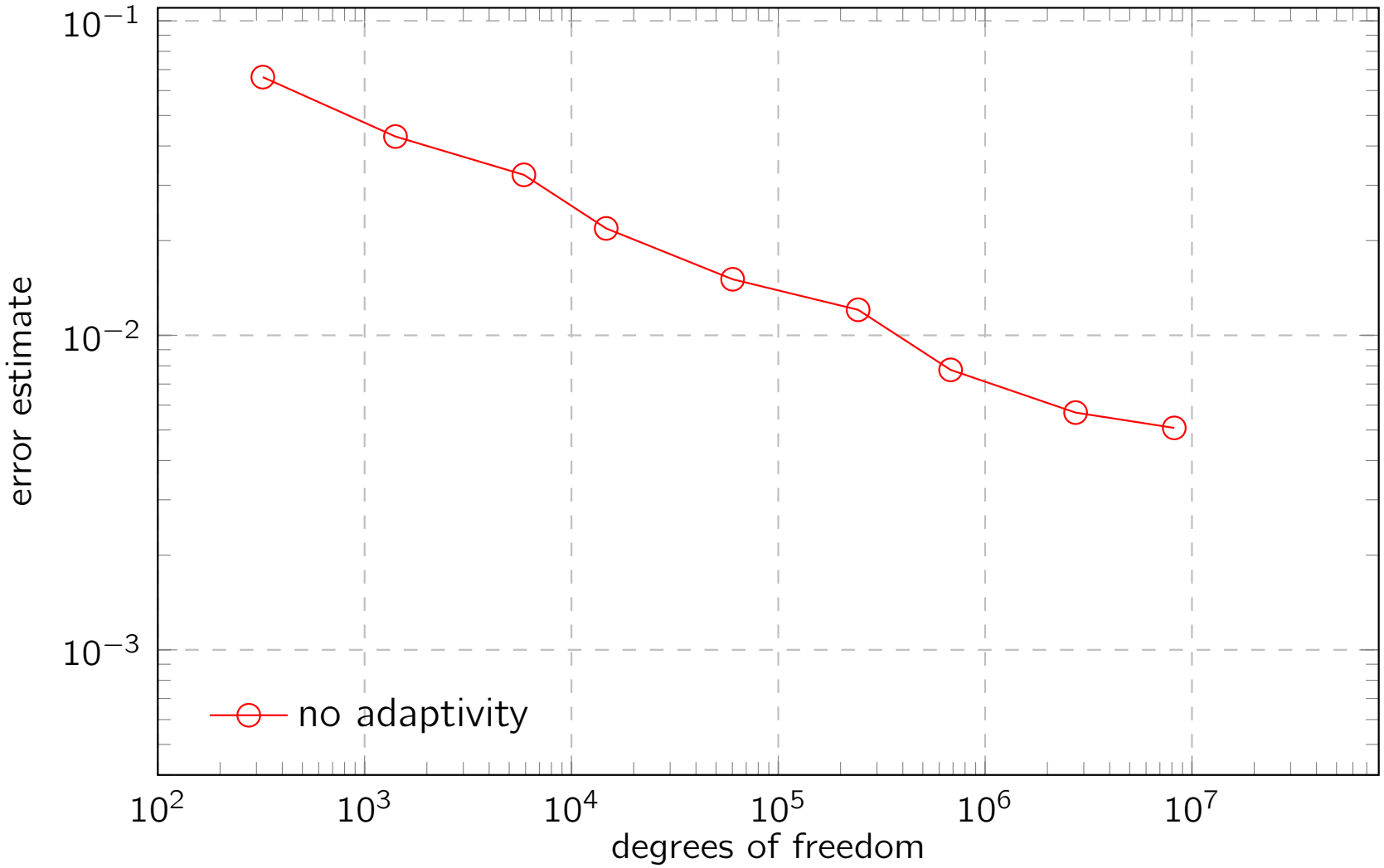




Experiment 1: advantages of adaptivity



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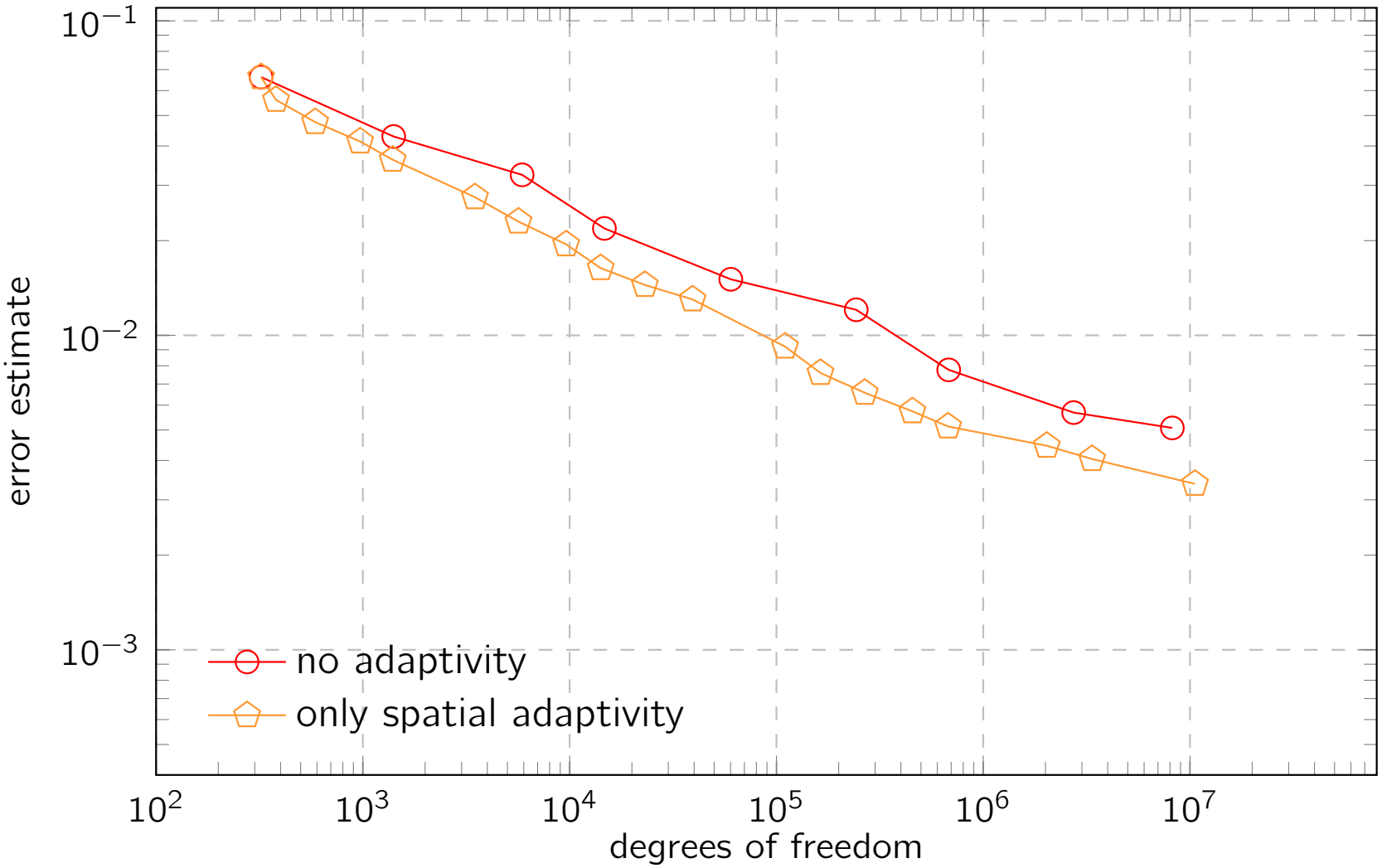

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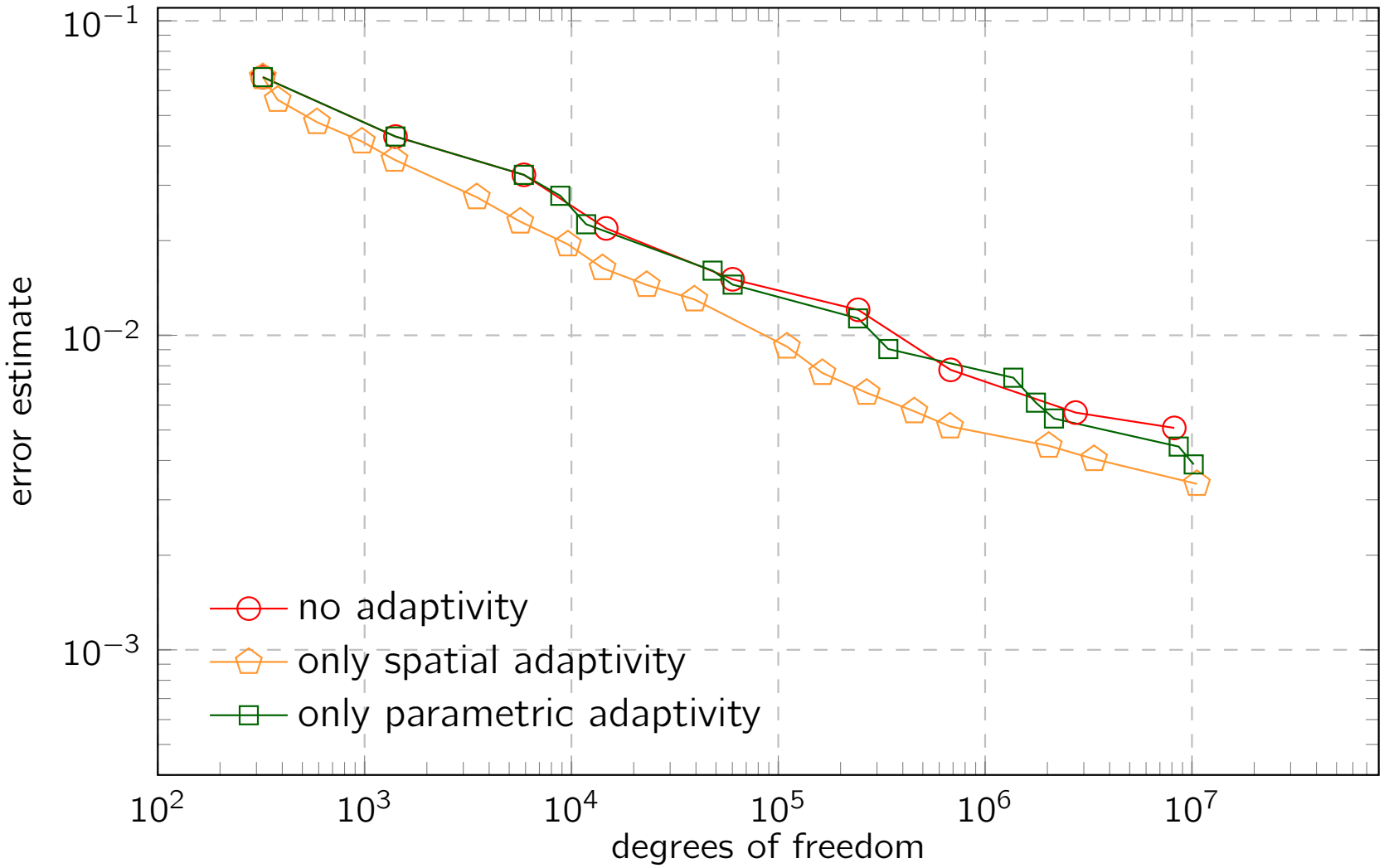

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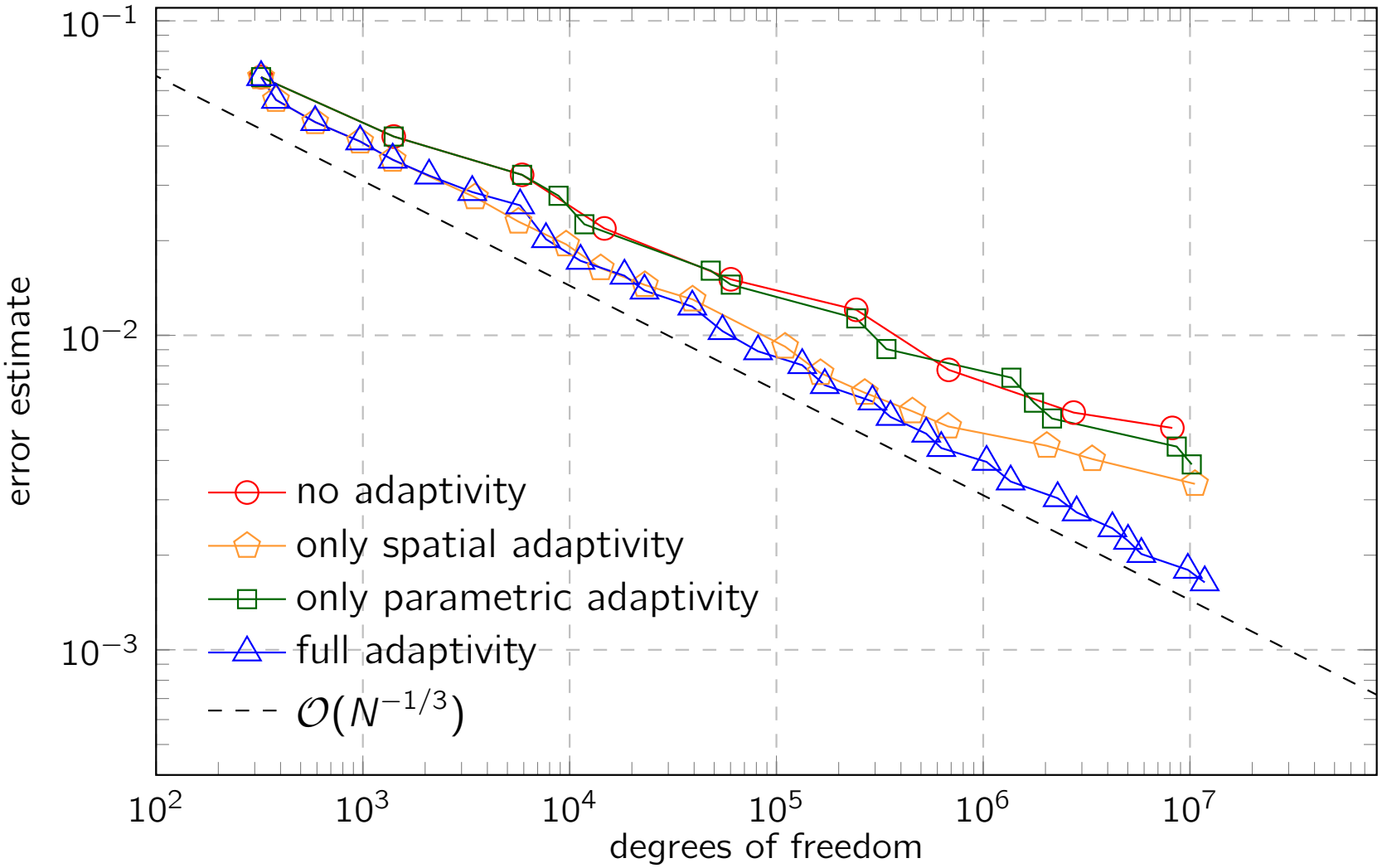

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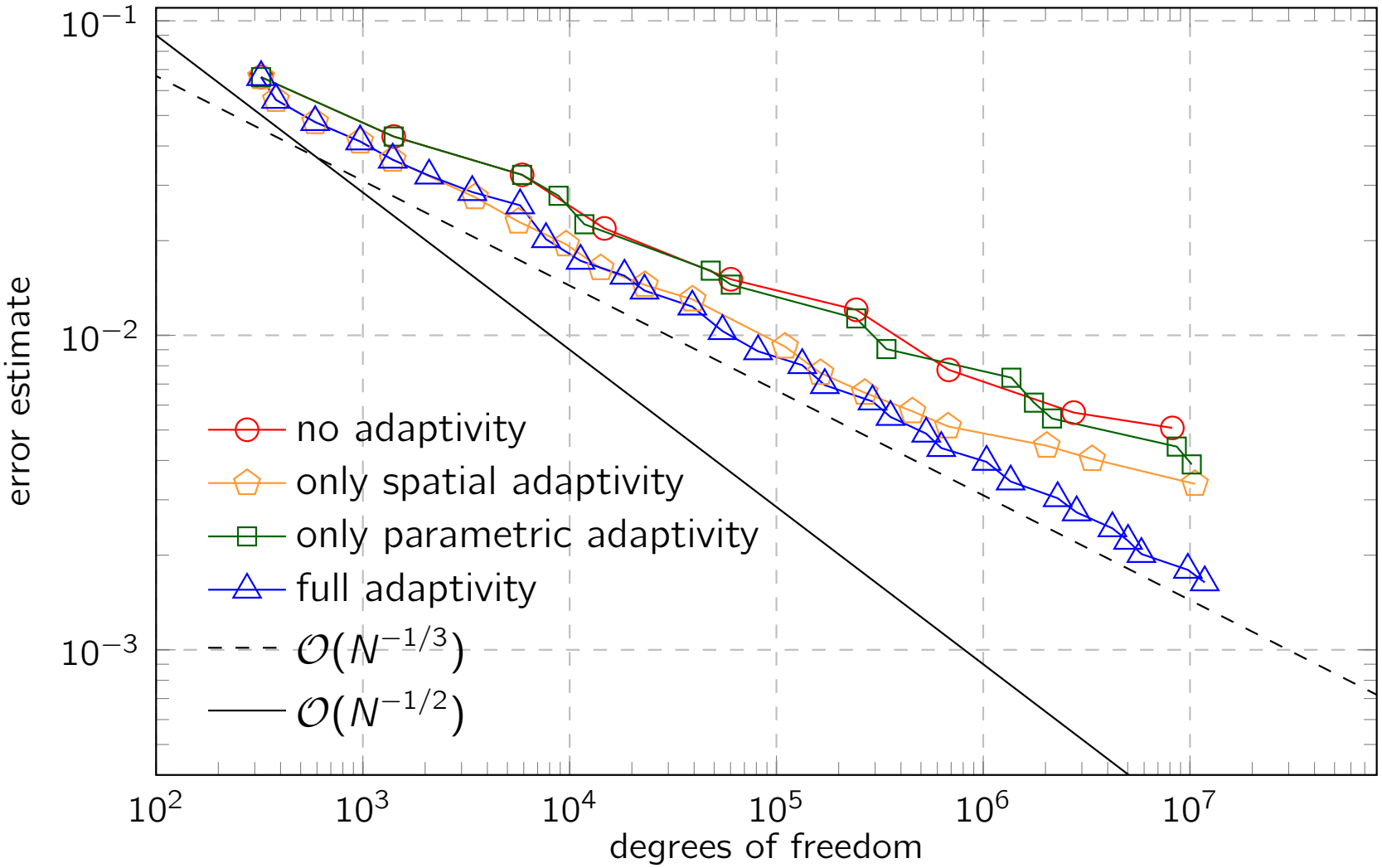

- $\mathcal{O}(N^{-1/2})$ $\rightsquigarrow$ optimal rate for parameter-free problem

Multilevel methods

- Idea $\rightsquigarrow$ use a hierarchy of approximations in physical domain
- Multilevel methods in UQ
  - Multilevel Monte Carlo
    [Giles '08], [Cliffe, Giles, Scheichl, Teckentrup '11], [Giles '15]
  - Multilevel quasi-Monte Carlo
    [Kuo, Schwab, Sloan '15]
  - Multilevel stochastic collocation
    [Teckentrup, Jantsch, Webster, Gunzburger '15]
  - Multilevel stochastic Galerkin FEM
    - theoretical benchmarks for convergence analysis
      [Cohen, DeVore, Schwab '10, '11], [Gittelson '13]
    - practical realisations
      [Eigel, Gittelson, Schwab, Zander '14], [Crowder, Powell, B. '19]
    - optimal convergence rates for practical realisations $\rightsquigarrow$ open problem...

Multilevel stochastic Galerkin FEM (ML-SGFEM)

- Idea $\rightsquigarrow$ use a hierarchy of approximations in physical domain
- SGFEM approximations $\rightsquigarrow$ finite (sparse) gPC expansions
  - $u_\bullet(x, \mathbf{y}) = \sum_{\nu \in \mathcal{P}_\bullet} u_{\bullet\nu}(x) P_\nu(\mathbf{y})$ with coefficients $u_{\bullet\nu} \in \mathbb{X}_\bullet = \mathcal{S}_0^1(\mathcal{T}_\bullet)$
  - $\mathbb{V}_\bullet := \mathbb{X}_\bullet \otimes \mathbb{P}_\bullet$, $\dim \mathbb{V}_\bullet = (\dim \mathbb{X}_\bullet) \times (\#\mathcal{P}_\bullet)$ $\rightsquigarrow$ single-level SGFEM

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- Multilevel discretisation in physical domain $\rightsquigarrow$ multilevel SGFEM
  - Hierarchy of meshes $\rightsquigarrow$ $\{\mathcal{T}_{\bullet\nu}\}_{\nu\in\mathcal{P}_\bullet}$
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  - Hierarchy of meshes $\rightsquigarrow$ $\{\mathcal{T}_{\bullet\nu}\}_{\nu\in\mathcal{P}_\bullet}$
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  - $\mathbb{V}_\bullet := \bigoplus_{\nu\in\mathcal{P}_\bullet} \left[\mathbb{X}_{\bullet\nu} \otimes \operatorname{span}\{P_\nu\}\right]$
  - $\dim \mathbb{V}_\bullet = \sum_{\nu\in\mathcal{P}_\bullet} \dim \mathbb{X}_{\bullet\nu}$

A posteriori error estimation in ML-SGFEM

[B., Praetorius, Ruggeri; SIAM/ASA JUQ '21 (to appear)]

- Extending *a posteriori* two-level/hierarchical error estimator to ML-SGFEM

$$\eta_\bullet^2 = (\text{estim. } \mathbb{X}\text{-error})^2 + (\text{estim. } \mathbb{P}\text{-error})^2 = \sum_{\nu\in\mathcal{P}_\bullet} \sum_{z\in\mathcal{N}_{\bullet,\nu}^+} \eta_\bullet^2(\nu, z) + \sum_{\nu\in\mathcal{Q}_\bullet} \eta_\bullet^2(\nu)$$

The goal is to prove $\eta_\bullet \simeq |||\, \widehat{u}_\bullet - u_\bullet \,|||$ $\rightsquigarrow$ What is $\widehat{u}_\bullet$ in the multilevel case?

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Single-level SGFEM: $\widehat{u}_\bullet \in \bigoplus_{\nu\in\mathcal{P}_\bullet} \big[\widehat{\mathbb{X}}_\bullet \otimes \mathrm{span}\{P_\nu\}\big] \oplus \bigoplus_{\nu\in\mathcal{Q}_\bullet} \big[\mathbb{X}_\bullet \otimes \mathrm{span}\{P_\nu\}\big]$

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Theorem 3

- $\eta_\bullet \simeq |||\, \widehat{u}_\bullet - u_\bullet \,|||$
- $|||\, u - u_\bullet \,||| \gtrsim \eta_\bullet$ (efficiency)
- saturation assumption $\Longrightarrow$ $|||\, u - u_\bullet \,||| \lesssim \eta_\bullet$ (reliability)

ML-SGFEM: challenges

ML-SGFEM: find $u_\bullet \in \mathbb{V}_\bullet = \bigoplus_{\nu \in \mathcal{P}_\bullet} \left[\mathbb{X}_{\bullet\nu} \otimes \mathrm{span}\{P_\nu\}\right]$ s.t. $B(u_\bullet, v) = F(v) \ \ \forall v \in \mathbb{V}_\bullet$.

- Implementation
  - Computation of non-square stiffness matrices associated with two different FE meshes
    Key observation: any two meshes are NVB refinements of the same initial mesh $\mathcal{T}_0$ $\rightsquigarrow$ exploiting the binary tree structure of the NVB refinement
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- Adaptive construction of $\mathbb{X}_{\bullet\nu}$ $\rightsquigarrow$ marking and refinement of FE meshes
- Analysis $\rightsquigarrow$ optimal convergence rate (?)

Marking strategy

- `MARK`: mark certain vertices $\mathcal{M}_{\ell\nu} \subseteq \mathcal{N}_{\ell\nu}^{+}$ $(\nu \in \mathcal{P}_\ell)$ and indices $\mathcal{R}_\ell \subseteq \mathcal{Q}_\ell$

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- **Idea** $\rightsquigarrow$ combined marking/enrichment of spatial and parametric components
  - Use Dörfler marking for

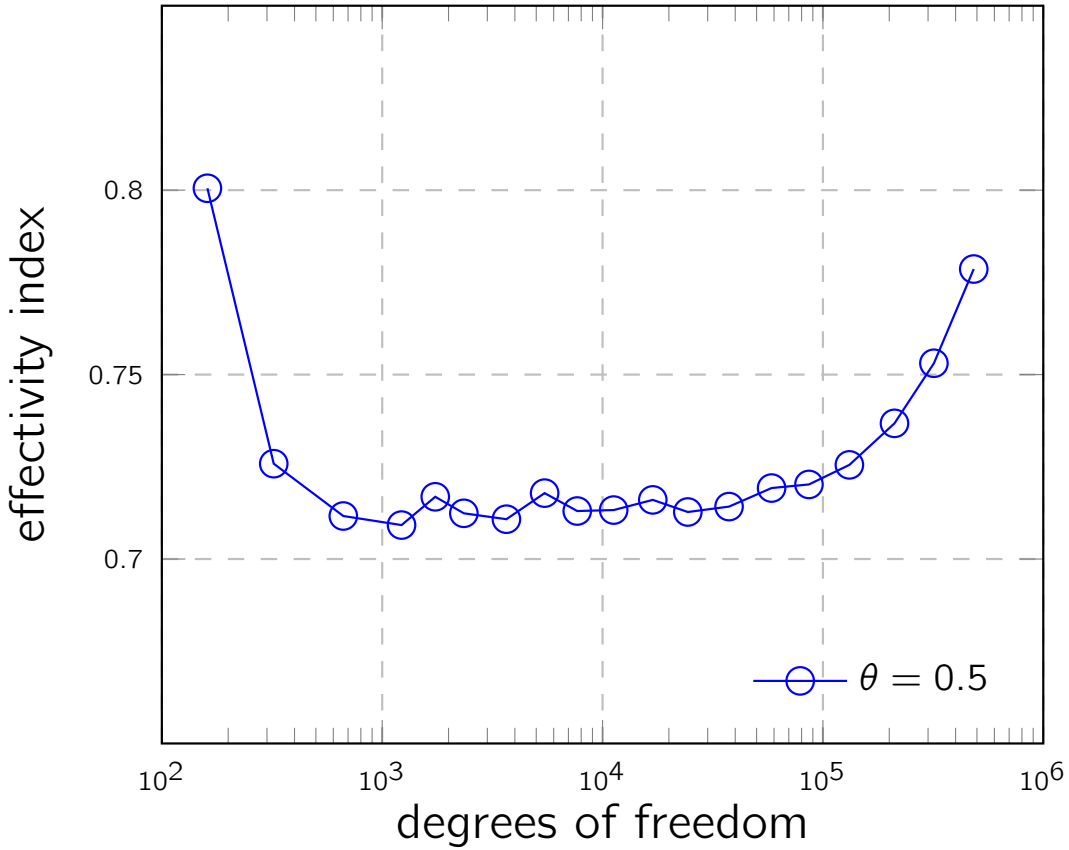
$$\{\eta_\ell(\nu, z);\ \nu \in \mathcal{P}_\ell,\ z \in \mathcal{N}_{\ell,\nu}^{+}\} \cup \{\eta_\ell(\nu);\ \nu \in \mathcal{Q}_\ell\}$$

with $\theta \in (0, 1]$ yields $\mathcal{M}_{\ell,\nu} \subseteq \mathcal{N}_{\ell,\nu}^{+}$ $(\nu \in \mathcal{P}_\ell)$ and $\mathcal{R}_\ell \subseteq \mathcal{Q}_\ell$ satisfying

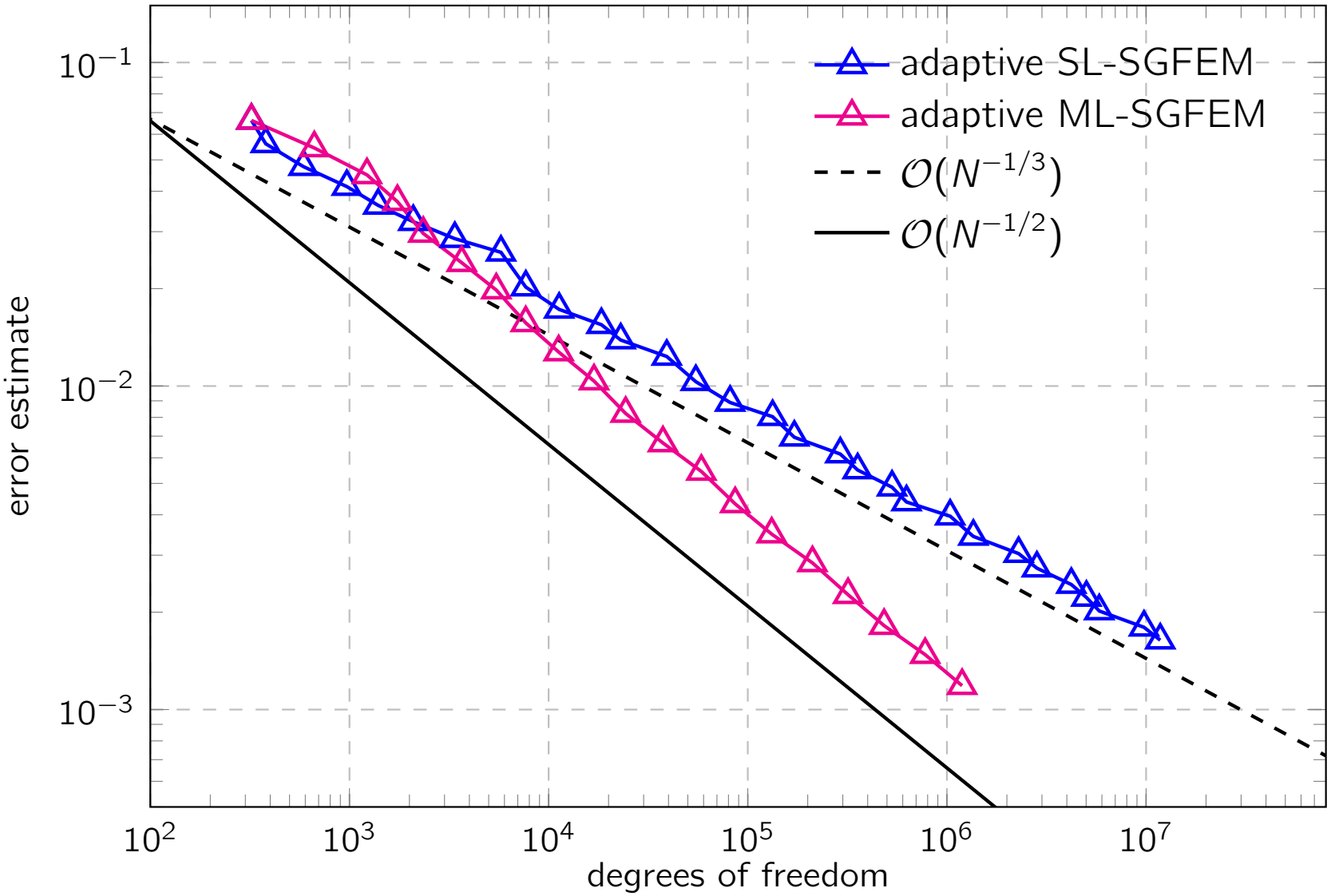
$$\theta\, \eta_\ell^2 \leq \sum_{\nu \in \mathcal{P}_\ell} \sum_{z \in \mathcal{M}_{\ell,\nu}} \eta_\ell^2(\nu, z) + \sum_{\nu \in \mathcal{R}_\ell} \eta_\ell^2(\nu)$$

Experiment 2: effectivity of the error estimation

$$\text{Effectivity index} := \frac{\text{error estimate}}{\text{reference error}} = \frac{\eta_\ell}{(|||\, u_{\mathrm{ref}} \,|||^2 - |||\, u_\ell \,|||^2)^{1/2}}.$$



Experiment 2: rate optimality of adaptive ML-SGFEM

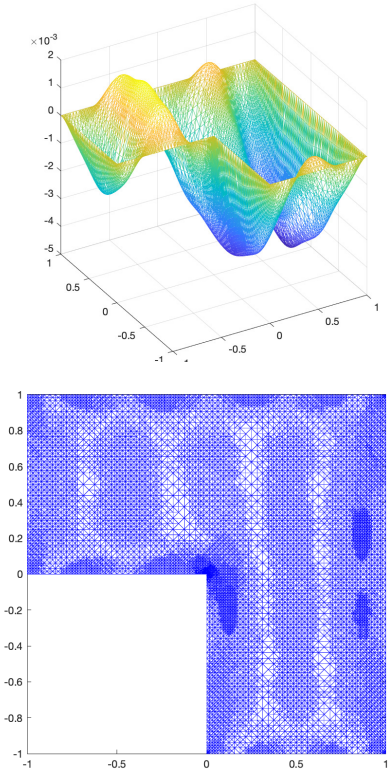


- $\mathcal{O}(N^{-1/2})$ $\rightsquigarrow$ optimal rate for parameter-free problem

Experiment 2: evolution of the index set

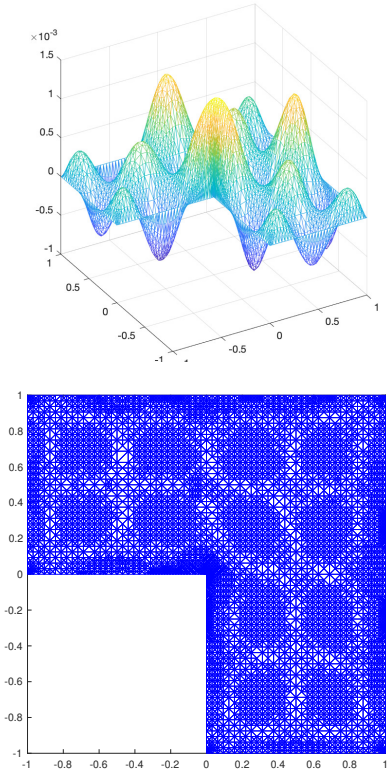
$\ell = 0$	$(0\ 0)$
$\ell = 1$	$(1\ 0)$
$\ell = 2$	$(0\ 1)$
	$(2\ 0)$
$\ell = 3$	$(0\ 0\ 1)$
	$(1\ 1\ 0)$
	$(3\ 0\ 0)$
$\ell = 4$	$(0\ 0\ 0\ 1)$
	$(1\ 0\ 1\ 0)$
$\ell = 5$	$(0\ 0\ 0\ 0\ 1)$
	$(2\ 1\ 0\ 0\ 0)$
$\ell = 6$	$(0\ 0\ 0\ 0\ 0\ 1)$
	$(1\ 0\ 0\ 1\ 0\ 0)$
	$(2\ 0\ 1\ 0\ 0\ 0)$
	$(0\ 2\ 0\ 0\ 0\ 0)$
	$(4\ 0\ 0\ 0\ 0\ 0)$
$\dots$	$\dots \quad \dots$

Experiment 2: locally refined meshes in ML-SGFEM



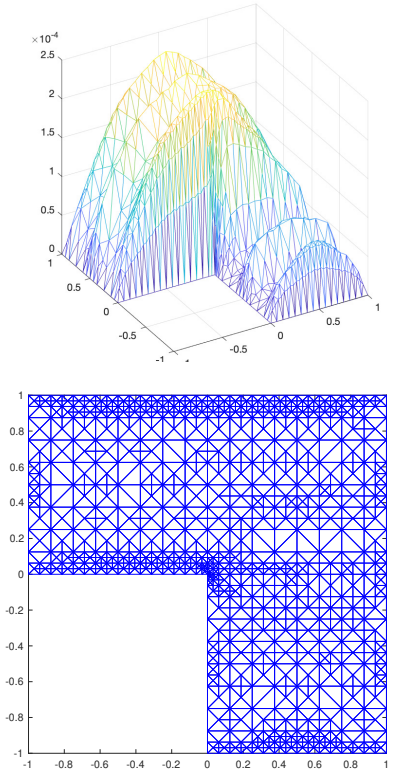
$\nu = (0\ 1)$

$\#\mathcal{T}_{\ell\nu} = 24664$



$\nu = (1\ 1)$

$\#\mathcal{T}_{\ell\nu} = 11550$



$\nu = (4\ 0)$

$\#\mathcal{T}_{\ell\nu} = 1809$

Optimal convergence of adaptive ML-SGFEM

[B., Praetorius, Ruggeri, IMANUM '21 (appeared online)]

- Concept of 'multilevel structure' $\rightsquigarrow$ $\mathbb{P}_\bullet = [\mathcal{P}_\bullet, (\mathcal{T}_{\bullet\nu})_{\nu\in\mathcal{P}_\bullet}]$
- Concept of 'multilevel refinement' $\rightsquigarrow$ $\mathbb{P}_\circ = \mathbb{REFINE}(\mathbb{P}_\bullet, \mathbb{M}_\bullet)$

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$$u \in \mathbb{A}_s \quad \Longleftrightarrow \quad \exists \left\{\mathbb{P}_\ell^\star\right\}_{\ell\in\mathbb{N}_0} \text{ such that } \|\!| u - u_\ell^\star |\!\| = \mathcal{O}\big((\dim \mathbb{V}_\ell^\star)^{-s}\big)$$

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Theorem 4 (convergence & rate optimality of adaptive ML-SGFEM)

- Saturation assumption $\Longrightarrow$ linear convergence for each $\theta \in (0, 1]$

$$\exists\, q \in (0,1) \text{ such that } \|\!| u - u_{\ell+1} |\!\| \le q \, \|\!| u - u_\ell |\!\| \quad \text{for all } \ell \in \mathbb{N}_0$$

- Strong saturation assumption $\Longrightarrow$ optimal convergence for sufficiently small θ

$$\text{If } s > 0 \text{ and } u \in \mathbb{A}_s, \text{ then } \sup_{\ell\in\mathbb{N}_0} (\#\mathbb{P}_\ell - \#\mathbb{P}_0 + 1)^s \, \|\!| u - u_\ell |\!\| \le C \|u\|_{\mathbb{A}_s}$$

Concluding remarks

What have we achieved?

- A *framework* for the design, analysis and implementation of adaptive algorithms for stochastic Galerkin FEM (single-level and multilevel)
  - Novel reliable and efficient a posteriori error estimates
  - Theoretically justified error reduction indicators to drive adaptivity
  - Convergence and rate optimality analysis
  - Efficient implementation and extensive testing
- Extensions (for single-level SGFEM so far)
  - Goal-oriented error estimation and adaptivity
    [B., Praetorius, Rocchi, Ruggeri; CMAME '19]
  - Problems with non-affine parametric data
    [B., Xu '20]
  - Parameter-dependent linear elasticity (incl. the incompressible limit case)
    [Khan, B., Powell, Silvester; Math. Comp. '21]

References & Software

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