Localization of dual norms, local stopping criteria, and fully adaptive solvers

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Outline

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1. Residuals and their dual norms
   - Laplace
   - Nonlinear Laplace
2. Localization dual norms
   - Local–global equivalence
   - Numerical results
3. Fully adaptive solvers
   - Setting
   - A posteriori guaranteed upper bound
   - Local stopping criteria, efficiency, and robustness
   - Applications
   - Numerical results
4. Conclusions and ongoing work

Outline

Residual and its dual norm for Laplacian, $u_h \in H_0^1(\Omega)$

The Laplace problem

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\begin{aligned}
-\Delta u &= f \quad && \text{in } \Omega, \\
u &= 0 \quad && \text{on } \partial\Omega
\end{aligned}
$$

- polytope $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Residual $\mathcal{R}(u_h) \in H^{-1}(\Omega)$

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v), \qquad v \in H_0^1(\Omega)$$ weak form. misfit

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Theorem (Equivalence energy error–dual norm of the residual)

Let $u_h \in H_0^1(\Omega)$. Then

$$\|\mathcal{R}(u_h)\|_{-1} = \|\nabla(u - u_h)\|.$$

Proof.

- residual and its dual norm definition

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- weak solution definition

$$(f, v) = (\nabla u, \nabla v)$$

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The nonconforming case, $u_h \notin H^1_0(\Omega)$

Theorem (Energy error in the nonconforming case)

Let $u_h \notin H^1_0(\Omega)$. Then

$$\|\nabla(u-u_h)\|^2 = \underbrace{\sup_{v\in H^1_0(\Omega);\, \|\nabla v\|=1} \{(f,v)-(\nabla u_h,\nabla v)\}^2}_{\|\mathcal{R}(u_h)\|_{-1},\ \text{dual norm of the residual}} + \underbrace{\min_{v\in H^1_0(\Omega)} \|\nabla(v-u_h)\|^2}_{\text{distance of } u_h \text{ to } H^1_0(\Omega)}.$$

Proof.

- define $s \in H^1_0(\Omega)$ by (projection)
$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H^1_0(\Omega)$$
- develop (Pythagoras)
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- example: p-Laplacian with $\boldsymbol{\sigma}(u,\nabla u) = |\nabla u|^{p-2}\nabla u$

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The nonlinear Laplace equation

The game

Is it possible to **localize** the dual norm of the residual

$$\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'} \approx \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}(u_h^{k,i})\|^q_{W_0^{1,p}(\omega_{\mathbf{a}})'} \right\}^{\frac{1}{q}} ?$$

- $\mathcal{V}_h$ vertices, $\omega_{\mathbf{a}}$ patches of elements of a partition $\mathcal{T}_h$ of Ω;
- the constant hidden in $\approx$ must not depend on p, Ω, and the regularity of u.

How to give tight and robust **computable bounds** on $\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'}$ on each Newton step k and algebraic step i?

How to **steer adaptively** (adaptive stopping criteria, adaptive mesh refinement) the inexact Newton solver?

How to take into account **nonconforming discretizations**?

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How to give tight and robust **computable bounds** on $\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'}$ on each Newton step k and algebraic step i?

How to **steer adaptively** (adaptive stopping criteria, adaptive mesh refinement) the inexact Newton solver?

How to take into account **nonconforming discretizations**?

The nonlinear Laplace equation

The game

Is it possible to **localize** the dual norm of the residual

$$\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'} \approx \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}(u_h^{k,i})\|^q_{W_0^{1,p}(\omega_{\mathbf{a}})'} \right\}^{\frac{1}{q}} ?$$

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Eisenstat and Walker (1994), Deuflhard (1996), Chaillou and Suri (2006, 2007), Kim (2007)

Outline

Outline

Localization dual norms

Setting

- $V := W_0^{1,p}(\Omega)$, $p > 1$, bounded linear functional $\mathcal{R} \in V'$
- localized energy space $V^{\mathbf{a}} := W_0^{1,p}(\omega_{\mathbf{a}})$ for $\mathbf{a} \in \mathcal{V}_h$
- restriction of $\mathcal{R}$ to $(V^{\mathbf{a}})'$ (zero extension of $v \in V^{\mathbf{a}}$),

$$\langle \mathcal{R}, v \rangle_{(V^{\mathbf{a}})', V^{\mathbf{a}}} := \langle \mathcal{R}, v \rangle_{V', V} \qquad v \in V^{\mathbf{a}}$$

$$\|\mathcal{R}\|_{(V^{\mathbf{a}})'} := \sup_{v \in V^{\mathbf{a}};\, \|\nabla v\|_{p,\omega_{\mathbf{a}}} = 1} \langle \mathcal{R}, v \rangle_{(V^{\mathbf{a}})', V^{\mathbf{a}}}$$

Theorem (Localization of $\|\mathcal{R}\|_{V'}$)

There holds

$$\|\mathcal{R}\|_{V'} \leq (d+1)^{\frac{1}{p}} C_{\mathrm{cont,PF}} \left\{ \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}} \quad \text{if } \langle \mathcal{R}, \psi_{\mathbf{a}} \rangle = 0 \;\; \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}},$$

$$\left\{ \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}} \leq (d+1)^{\frac{1}{q}} \|\mathcal{R}\|_{V'}.$$

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Localization of the dual residual norm

Upper bound (needs vanishing lowest modes).

- partition of unity, the linearity of $\mathcal{R}$, orthogonality wrt $\psi_{\mathbf{a}}$:

$$\langle \mathcal{R}, v\rangle = \sum_{\mathbf{a}\in\mathcal{V}_h} \langle \mathcal{R}, \psi_{\mathbf{a}} v\rangle = \sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} \langle \mathcal{R}, \psi_{\mathbf{a}}(v - \Pi_{0,\omega_{\mathbf{a}}} v)\rangle + \sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{ext}}} \langle \mathcal{R}, \psi_{\mathbf{a}} v\rangle$$

- stability:

$$\|\nabla(\psi_{\mathbf{a}}(v - \Pi_{0,\omega_{\mathbf{a}}} v))\|_{p,\omega_{\mathbf{a}}} \leq C_{\mathrm{cont,PF}} \|\nabla v\|_{p,\omega_{\mathbf{a}}}$$

- Hölder inequality:

$$\langle \mathcal{R}, v\rangle \leq C_{\mathrm{cont,PF}} \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}} \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla v\|^p_{p,\omega_{\mathbf{a}}} \right\}^{\frac{1}{p}}$$

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Localization of the dual residual norm

Lower bound (unconditioned).

- p-Laplacian lifting of the residual on the patch $\omega_{\mathbf{a}}$:
  $r^{\mathbf{a}} \in V^{\mathbf{a}} = W_0^{1,p}(\omega_{\mathbf{a}})$ such that
$$(|\nabla r^{\mathbf{a}}|^{p-2}\nabla r^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, v\rangle \qquad \forall v \in V^{\mathbf{a}}$$
- energy equality:
$$\|\nabla r^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p = (|\nabla r^{\mathbf{a}}|^{p-2}\nabla r^{\mathbf{a}}, \nabla r^{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, r^{\mathbf{a}}\rangle = \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q$$
- setting $r := \sum_{\mathbf{a}\in\mathcal{V}_h} r^{\mathbf{a}} \in V$:
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- overlapping of the patches:
$$\|\nabla r\|_p^p \le (d+1)^{p-1}\sum_{\mathbf{a}\in\mathcal{V}_h}\|\nabla r^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p$$

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Outline

Numerical results

Model problems

- p-Laplacian

$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega,$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

- $\Omega = (0,1)\times(0,1)$ and, for $p = 1.5$ and 10,

$$u(x,y) = -\frac{p-1}{p}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

- $\Omega = (-1,1)\times(-1,1)\setminus[0,1]\times[-1,0]$ and, for $p = 4$,

$$u(r,\theta) = r^{\frac{7}{8}}\sin(\theta\tfrac{7}{8})$$

- three successive uniformly refined meshes

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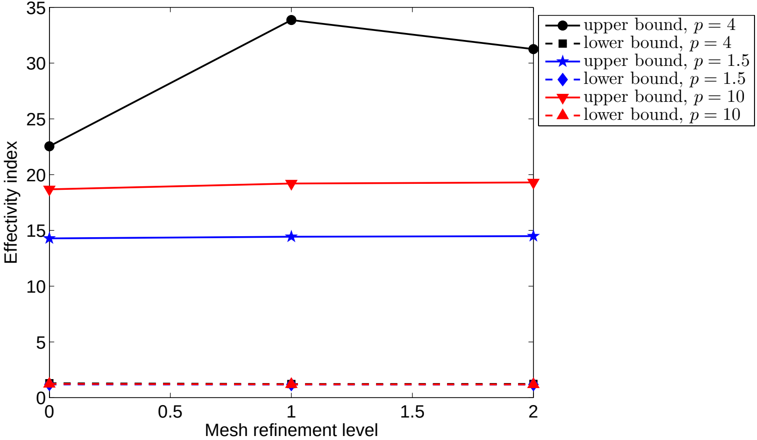
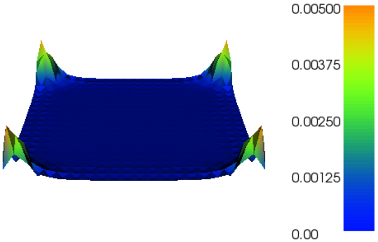
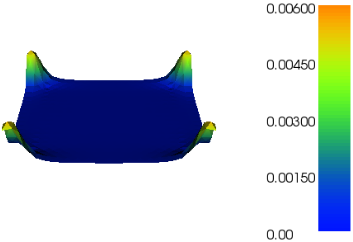
Effectivity indices of the localization bounds



Local and global residual distributions, $p = 1.5$

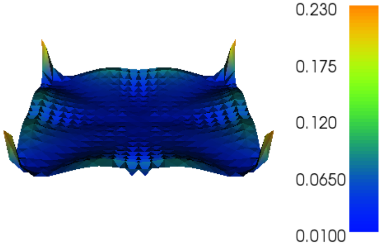


Local

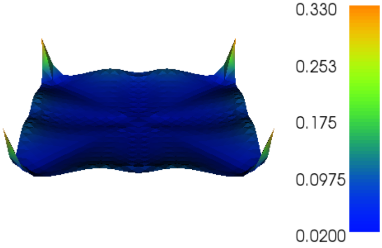


Global

Local and global residual distributions, $p = 10$

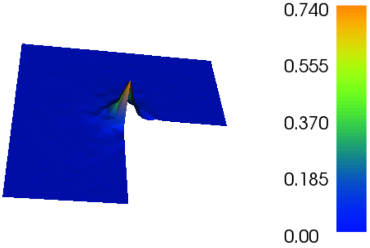


Local



Global

Local and global residual distributions, $p = 4$



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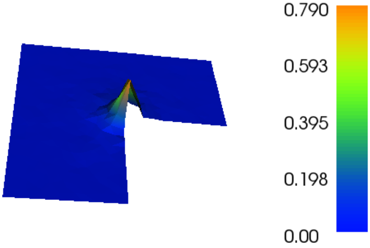

Global

Outline

Outline

Abstract assumptions

Numerical approximation

- simplicial mesh $\mathcal{T}_h$, linearization step k, algebraic step i
- $u_h^{k,i} \in V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\} \not\subset V$

Assumption A (Total flux reconstruction)

There exists $\boldsymbol{\sigma}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ and $\rho_h^{k,i} \in L^q(\Omega)$ such that

$$\nabla \cdot \boldsymbol{\sigma}_h^{k,i} = f_h - \underbrace{\rho_h^{k,i}}_{\text{algebraic remainder}}.$$

Assumption B (Discretization, linearization, and alg. fluxes)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ such that

(i) $\boldsymbol{\sigma}_h^{k,i} = \mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i}$;

(ii) *as the linear solver converges,* $\|\mathbf{a}_h^{k,i}\|_q \to 0$;

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(iii) *as the nonlinear solver converges,* $\|\mathbf{l}_h^{k,i}\|_q \to 0$.

Abstract assumptions

Numerical approximation

- simplicial mesh $\mathcal{T}_h$, linearization step k, algebraic step i
- $u_h^{k,i} \in V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\} \not\subset V$

Assumption A (Total flux reconstruction)

There exists $\boldsymbol{\sigma}_h^{k,i} \in \mathbf{H}^q(\mathrm{div},\Omega)$ and $\rho_h^{k,i} \in L^q(\Omega)$ such that

$$\nabla\cdot\boldsymbol{\sigma}_h^{k,i} = f_h - \underbrace{\rho_h^{k,i}}_{\text{algebraic remainder}} .$$

Assumption B (Discretization, linearization, and alg. fluxes)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ such that

(i) $\boldsymbol{\sigma}_h^{k,i} = \mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i}$;

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Outline

Estimate distinguishing error components

Theorem (Estimate distinguishing different error components)

Let

- *$u \in V$ be the weak solution,*
- *$u_h^{k,i} \in V(\mathcal{T}_h)$ be arbitrary,*
- *Assumptions A and B hold.*

Then there holds

$$\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'} + \mathrm{NC} \le \eta_{\mathrm{disc}}^{k,i} + \underbrace{\eta_{\mathrm{lin}}^{k,i}}_{\|\mathbf{l}_h^{k,i}\|_q} + \underbrace{\eta_{\mathrm{alg}}^{k,i}}_{\|\mathbf{a}_h^{k,i}\|_q} + \underbrace{\eta_{\mathrm{rem}}^{k,i}}_{h_\Omega \|\rho_h^{k,i}\|_q} + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}},$$

with $\eta_{\cdot}^{k,i} := \left\{\sum_{K\in\mathcal{T}_h} (\eta_{\cdot,K}^{k,i})^q\right\}^{1/q}$ and

$$\eta_{\mathrm{disc},K}^{k,i} := 2^{\frac{1}{p}} \left(\|\overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) + \mathbf{d}_h^{k,i}\|_{q,K} + \left\{ \sum_{e\in\mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \right\}^{\frac{1}{q}} \right).$$

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Outline

Stopping criteria and efficiency

Global stopping criteria (≈ Becker, Johnson, and Rannacher (1995), Arioli (2000's))

$$\eta_{\rm rem}^{k,i} \le \gamma_{\rm rem} \max\{\eta_{\rm disc}^{k,i}, \eta_{\rm lin}^{k,i}, \eta_{\rm alg}^{k,i}\},$$
$$\eta_{\rm alg}^{k,i} \le \gamma_{\rm alg} \max\{\eta_{\rm disc}^{k,i}, \eta_{\rm lin}^{k,i}\}, \qquad \gamma_{\rm rem}, \gamma_{\rm alg}, \gamma_{\rm lin} \approx 0.1$$
$$\eta_{\rm lin}^{k,i} \le \gamma_{\rm lin} \eta_{\rm disc}^{k,i}$$

Local stopping criteria

- stop whenever:

$$\eta_{{\rm rem},K}^{k,i} \le \gamma_{{\rm rem},K} \max\{\eta_{{\rm disc},K}^{k,i}, \eta_{{\rm lin},K}^{k,i}, \eta_{{\rm alg},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{{\rm alg},K}^{k,i} \le \gamma_{{\rm alg},K} \max\{\eta_{{\rm disc},K}^{k,i}, \eta_{{\rm lin},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{{\rm lin},K}^{k,i} \le \gamma_{{\rm lin},K} \eta_{{\rm disc},K}^{k,i} \qquad \forall K \in \mathcal{T}_h$$

- $\gamma_{{\rm rem},K}, \gamma_{{\rm alg},K}, \gamma_{{\rm lin},K} \approx 0.1$

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$$\begin{aligned}\eta_{\mathrm{rem}}^{k,i} &\le \gamma_{\mathrm{rem}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}, \eta_{\mathrm{alg}}^{k,i}\},\\ \eta_{\mathrm{alg}}^{k,i} &\le \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\},\\ \eta_{\mathrm{lin}}^{k,i} &\le \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}\end{aligned} \qquad \gamma_{\mathrm{rem}}, \gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}} \approx 0.1$$

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- $\gamma_{\mathrm{rem},K}, \gamma_{\mathrm{alg},K}, \gamma_{\mathrm{lin},K} \approx 0.1$

Assumptions for efficiency

Assumption C (Piecewise polynomials, meshes, quadrature)

The approximation $u_h^{k,i}$ is piecewise polynomial. The meshes $\mathcal{T}_h$ are shape-regular. The quadrature error is negligible.

Assumption D (Approximation property)

For all $K \in \mathcal{T}_h$, there holds

$$\|\overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) + \mathbf{d}_h^{k,i}\|_{q,K} \leq C \Bigg\{ \sum_{K' \in \mathfrak{T}_K} h_{K'}^q \|f + \nabla \cdot \overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i})\|_{q,K'}^q + \sum_{e \in \mathfrak{E}_K^{\mathrm{int}}} h_e \|[\![\overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) \cdot \mathbf{n}_e]\!]\|_{q,e}^q + \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \Bigg\}^{\frac{1}{q}}.$$

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Global efficiency

Theorem (Global efficiency)

Let the Assumptions C and D be satisfied. Let the global stopping criteria hold. Then,

$$\eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} \le C\Big(\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'} + \mathrm{NC}\Big),$$

where C is independent of σ and q.

Theorem (Local efficiency)

*Let the Assumptions C and D be satisfied. Let the **local** stopping criteria hold. Then, for all $K \in \mathcal{T}_h$,*

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Outline

Nonconforming finite elements for the p-Laplacian

Discretization

Find $u_h \in V_h$ such that

$$(\boldsymbol{\sigma}(\nabla u_h), \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $\boldsymbol{\sigma}(\nabla u_h) = |\nabla u_h|^{p-2} \nabla u_h$
- $V_h \not\subset V$ the Crouzeix–Raviart space
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

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Linearization

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Find $u_h^k \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^k), \nabla \psi_e) = (f, \psi_e) \qquad \forall e \in \mathcal{E}_h^{\text{int}}.$$

- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2}\boldsymbol{\xi}$$

- Newton linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2}\boldsymbol{\xi} + (p-2)|\nabla u_h^{k-1}|^{p-4} (\nabla u_h^{k-1} \otimes \nabla u_h^{k-1})(\boldsymbol{\xi} - \nabla u_h^{k-1})$$

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$$\mathbb{A}^{k-1} U^k = F^{k-1}$$

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Algebraic solution

Algebraic solution

Find $u_h^{k,i} \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i}), \nabla \psi_e) = (f, \psi_e) - R_e^{k,i} \qquad \forall e \in \mathcal{E}_h^{\text{int}}.$$

- algebraic residual vector $R^{k,i} = \{R_e^{k,i}\}_{e \in \mathcal{E}_h^{\text{int}}}$
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Flux reconstructions

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$)

For all $K \in \mathcal{T}_h$,

$$(\mathbf{d}_h^{k,i}+\mathbf{l}_h^{k,i})|_K := -\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i})|_K + \frac{f|_K}{d}(\mathbf{x}-\mathbf{x}_K) - \sum_{e \in \mathcal{E}_K} \frac{R_e^{k,i}}{d|D_e|}(\mathbf{x}-\mathbf{x}_K)|_{K_e},$$

where $R_e^{k,i} = (f, \psi_e) - (\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i}), \nabla \psi_e) \quad \forall e \in \mathcal{E}_h^{\mathrm{int}}$.

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Set $\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ for (adaptively chosen) $\nu > 0$ additional algebraic solvers steps; $R^{k,i+\nu} \rightsquigarrow \rho_h^{k,i}$.

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Verification of the assumptions

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$

Lemma (Assumptions C and D)

Assumptions C and D hold.

Comments

- quadrature error is zero
- $\mathbf{d}_h^{k,i}$ is close to $\sigma(\nabla u_h^{k,i})$: approximation properties of the Raviart–Thomas–Nédélec spaces

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Summary

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to D verified

Summary

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

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Outline

Numerical experiment I

Model problem

- p-Laplacian

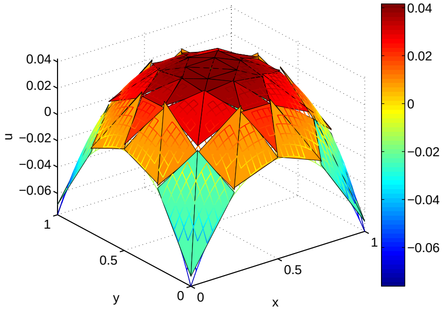
$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega,$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

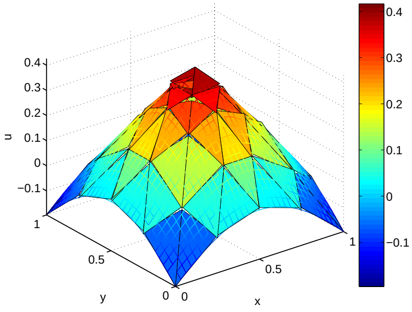
$$u(x,y) = -\tfrac{p-1}{p}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \tfrac{p-1}{p}\left(\tfrac{1}{2}\right)^{\frac{p}{p-1}}$$

- tested values $p = 1.5$ and 10
- Crouzeix–Raviart nonconforming finite elements

Analytical and approximate solutions

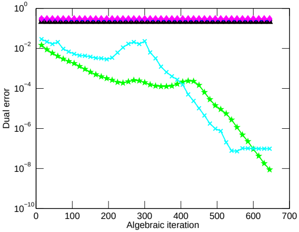


Case $p = 1.5$

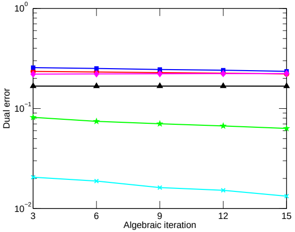


Case $p = 10$

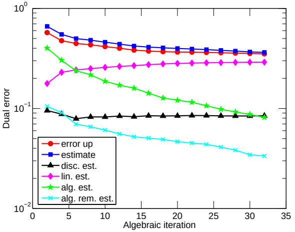
Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step.



Newton

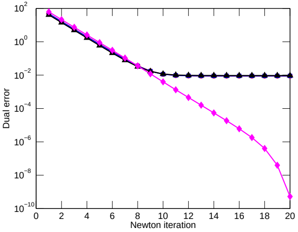


inexact Newton

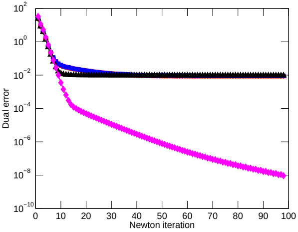


ad. inexact Newton

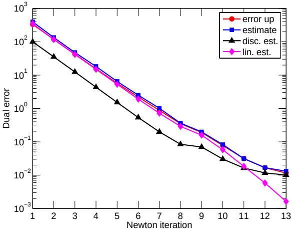
Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh



Newton

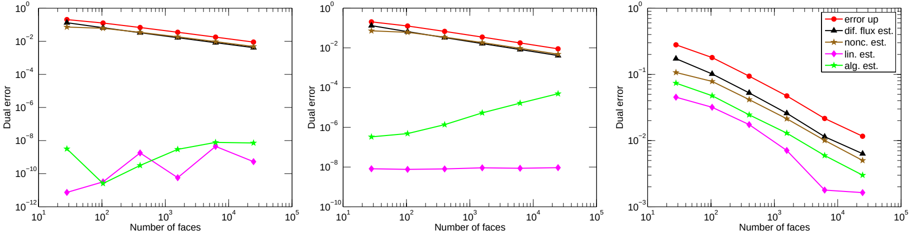


inexact Newton



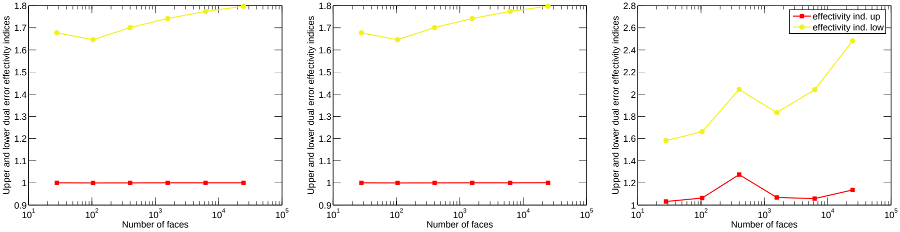
ad. inexact Newton

Error and estimators, $p = 10$



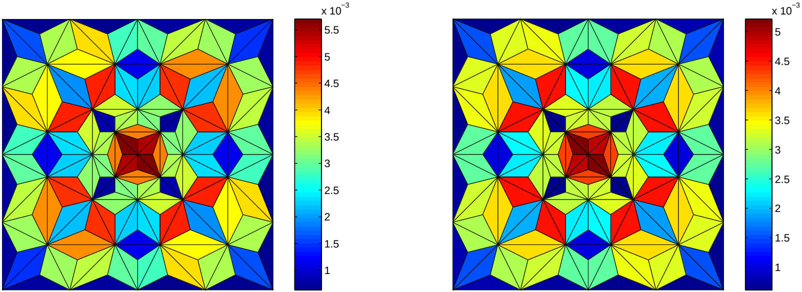
Newton inexact Newton ad. inexact Newton

Effectivity indices, $p = 10$



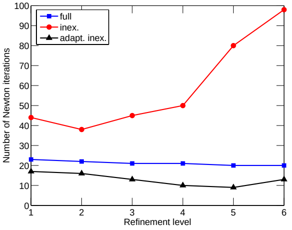
Newton | inexact Newton | ad. inexact Newton

Error distribution, $p = 10$

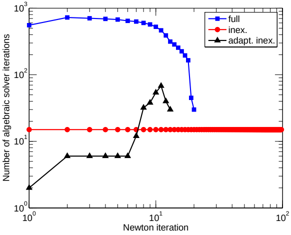


Estimated error distribution   Exact error distribution

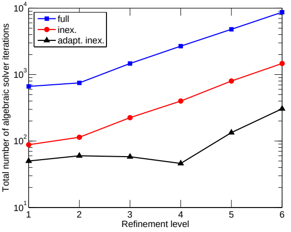
Newton and algebraic iterations, $p = 10$



Newton it. / refinement

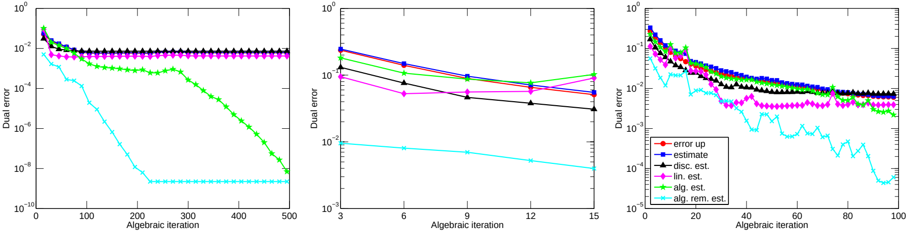


alg. it. / Newton step



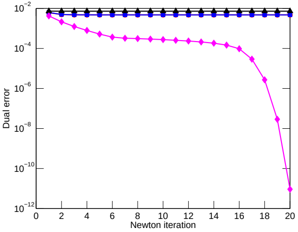
alg. it. / refinement

Error and estimators as a function of CG iterations, $p = 1.5$, 6th level mesh, 1st Newton step.

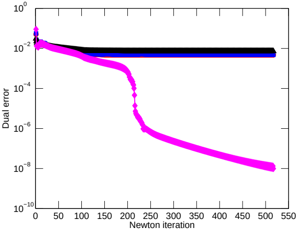


Newton inexact Newton ad. inexact Newton

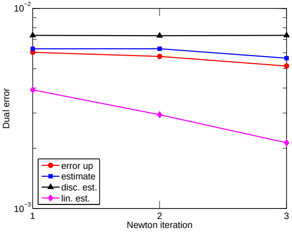
Error and estimators as a function of Newton iterations, $p = 1.5$, 6th level mesh



Newton



inexact Newton



ad. inexact Newton

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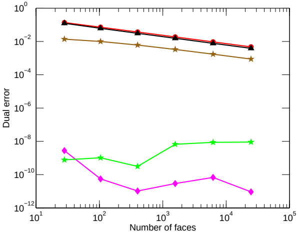
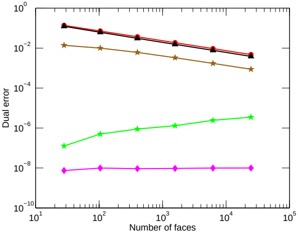
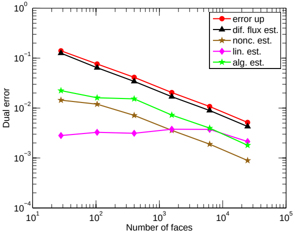


Newton

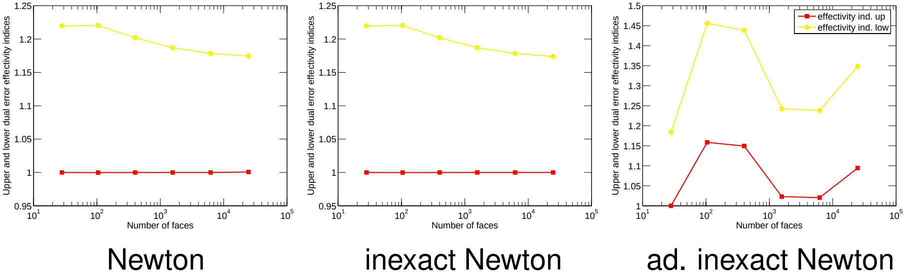


inexact Newton

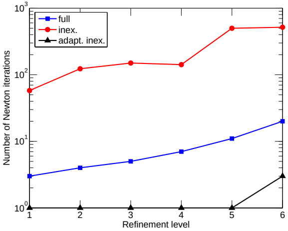


ad. inexact Newton

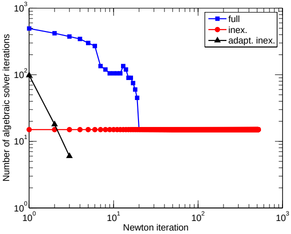
Effectivity indices, $p = 1.5$



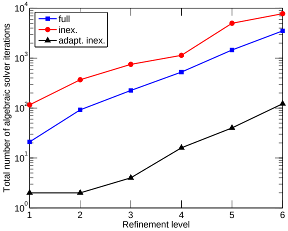
Newton and algebraic iterations, $p = 1.5$



Newton it. / refinement



alg. it. / Newton step



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Numerical experiment II

Model problem

- p-Laplacian

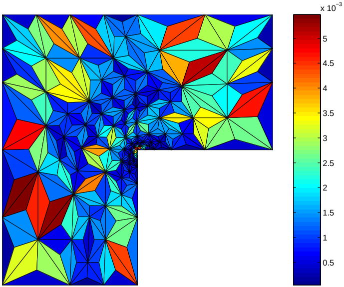
$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega,$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

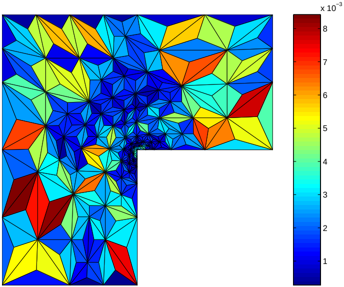
$$u(r,\theta) = r^{\frac{7}{8}} \sin(\theta \tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))
- Crouzeix–Raviart nonconforming finite elements

Error distribution on an adaptively refined mesh

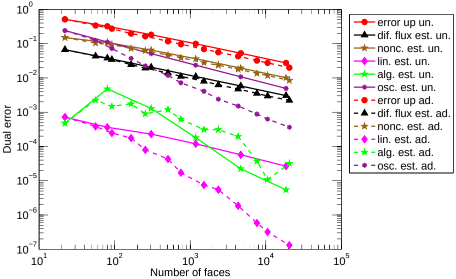


Estimated error distribution



Exact error distribution

Estimated and actual errors and the effectivity index



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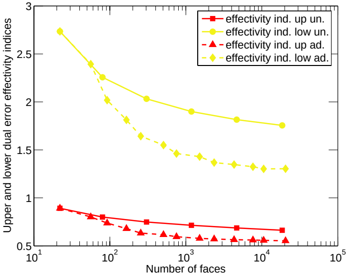
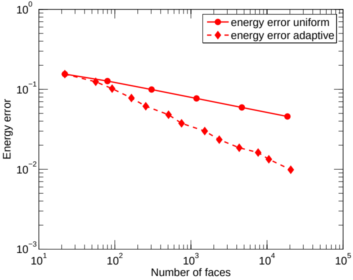
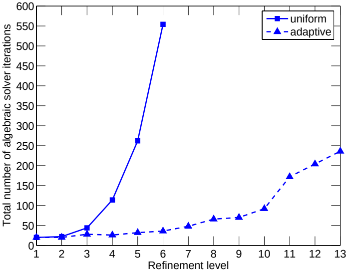


Effectivity index

Energy error and overall performance



Energy error



Overall performance

Outline

Conclusions and future directions

Conclusions

- dual residual norms are localizable
- local stopping criteria then lead to local efficiency
- concept of full adaptivity (linear solver, nonlinear solver, mesh)

Ongoing work

- multigrid as a linear solver
- convergence and optimality

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Bibliography

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- Blechta J., Málek J., Vohralík M., Localization of $W^{-1,q}$ norms for robust local a posteriori efficiency, to be submitted.

Thank you for your attention!