Adaptive wavelet methods: Quantitative improvements and extensions

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Contents

- Adaptive wavelet methods for solving well-posed operator equations with *symmetric, coercive* Fréchet derivatives
- An efficient approximate residual evaluation for 1st order systems
- Adaptive wavelet methods for solving *general* well-posed operator equations: Nonlinear least squares
- Time evolution problems: Simultaneous space-time variational formulations of parabolic problems and (N)SE

Well-posed op. eqs.

For $\mathcal{X}$ (real) sep. Hilbert space, let

- $F : \mathcal{X} \supset \mathrm{dom}(F) \to \mathcal{X}'$,
- F cont. Fréchet diff. in neighb. of a sol u of $\boxed{F(u) = 0,}$
- $DF(u) \in \mathcal{L}\mathrm{is}(\mathcal{X}, \mathcal{X}')$, $DF(u) = DF(u)' > 0$,

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Ex.

- $\Omega \subset \mathbb{R}^d$, $d \leq 3$, $\mathcal{X} = H_0^1(\Omega)$, $F(u)(v) = \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v + u^3 v - f v \, dx$
- $F(u)(v) = \frac{1}{4\pi} \int_{\partial\Omega} \left\{ \int_{\partial\Omega} \frac{(u(y)-u(x))(v(y)-v(x))}{|x-y|^3} \, dy - v(x) f(x) \right\} dx$,
  $\Omega \subset \mathbb{R}^3$, $\mathcal{X} = H^{\frac{1}{2}}(\partial\Omega)/\mathbb{R}$ (hypersingular boundary integral equation).

Reformulation as a countable set of coupled scalar eqs

Let $\Psi = \{\psi_\lambda \colon \lambda \in \nabla\}$ **Riesz basis** for $\mathcal{X}$, i.e., **synthesis operator**,

$$\mathcal{F} : \mathbf{c} \mapsto \mathbf{c}^\top \Psi := \sum_{\lambda \in \nabla} c_\lambda \psi_\lambda \in \mathcal{L}\mathrm{is}(\ell_2(\nabla), \mathcal{X}),$$

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Then with $\mathbf{F} = \mathcal{F}' F \mathcal{F} : \ell_2(\Lambda) \supset \mathrm{dom}(\mathbf{F}) \to \ell_2(\Lambda)$, equiv. form.

$$\boxed{\mathbf{F}(\mathbf{u}) = 0},$$

where $\mathbf{u} := \mathcal{F}^{-1} u$.

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Norm on $\ell_2(\nabla)$ will be denoted as $\|\cdot\|$.
$\|\mathbf{u} - \mathbf{w}\| \eqsim \|u - \mathcal{F}\mathbf{w}\|_{\mathscr{X}}$.

Adaptive wavelet Galerkin method

In its original form introduced by [Cohen, Dahmen, DeVore '01, 02]

Alg (awgm).

% *Let* $\mathbf{U} \subset \ell_2(\Lambda)$ *be a neigh. of* $\mathbf{u}$, $\mu \in (0,1]$, *finite* $\Lambda_0 \subset \nabla$.

`for` $i = 0, 1, \ldots$ `do`

solve $\mathbf{u}_i \in \mathbf{U}$ *with* $\operatorname{supp} \mathbf{u}_i \subseteq \Lambda_i$ *s.t.* $\mathbf{F}(\mathbf{u}_i)|_{\Lambda_i} = 0$

determine a smallest $\Lambda_{i+1} \supset \Lambda_i$ *s.t.* $\|\mathbf{F}(\mathbf{u}_i)|_{\Lambda_{i+1}}\| \geq \mu \|\mathbf{F}(\mathbf{u}_i)\|$

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Thm (convergence). $\exists \alpha < 1$ *s.t. when* $\mathbf{U}$ *and* $\inf_{\mathbf{v} \in \ell_2(\Lambda_0)} \|\mathbf{u} - \mathbf{v}\|$ *suff. small,* $\|\mathbf{u} - \mathbf{u}_i\| \lesssim \alpha^i \|\mathbf{u} - \mathbf{u}_0\|$.

For affine $\mathbf{F}$, use $|||\mathbf{u} - \mathbf{u}_{i+1}|||^2 = |||\mathbf{u} - \mathbf{u}_i|||^2 - |||\mathbf{u}_{i+1} - \mathbf{u}_i|||^2$, and saturation $|||\mathbf{u}_{i+1} - \mathbf{u}_i||| \gtrsim |||\mathbf{u} - \mathbf{u}_i|||$ by 'bulk chasing'. Perturb arg. for non-affine.

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Def (approx. class). For $s > 0$,

$$\mathcal{A}^s := \Big\{ \mathbf{u} \in \ell_2(\nabla) \colon \|\mathbf{u}\|_{\mathcal{A}^s} := \sup_{N \in \mathbb{N}} N^s \inf_{\{\mathbf{w} \colon \# \operatorname{supp} \mathbf{w} \leq N\}} \|\mathbf{u} - \mathbf{w}\| < \infty \Big\}.$$

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Thm (optimal rate). *If* μ *is suff. small, then if* $\mathbf{u} \in \mathcal{A}^s$,

$$(\# \operatorname{supp} \mathbf{u}_i)^s \|\mathbf{u} - \mathbf{u}_i\| \lesssim 1.$$

Practical awgm

Thm. *With* **approx.** *eval. of* $\mathbf{F}(\mathbf{u}_i)$ *with* **rel. tolerance** $\delta > 0$ *(suff. small but fixed),* **awgm** *also converges with optimal rate.*

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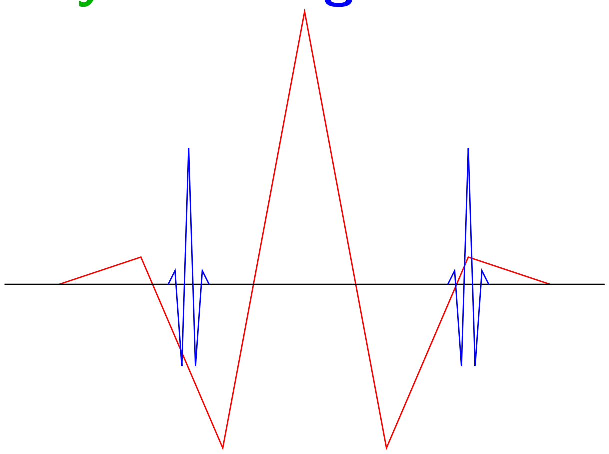
Such bases for the common Sob. spaces are available on general polygonal domains and consist of piecewise polynomial wavelets. Wavelet ψ_λ on 'level' $|\lambda| \in I\!N$ has $\operatorname{diam}\operatorname{supp}\psi_\lambda \eqsim 2^{-|\lambda|}$.



Usual residual evaluation ([CDD01])

For $F(u) = Au - f$, approximate both $\mathcal{F}' A \mathcal{F} \mathbf{u}_i$ and $\mathcal{F}' f$ **separately** within **absolute** tolerance $\frac{1}{2}\delta \|\mathbf{u} - \mathbf{u}_i\|$.

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Ex (Poisson)**.** Terms read as $\left[\int_\Omega \mathbf{grad}\,\Psi \cdot \mathbf{grad}\,\Psi\right]\mathbf{u}_i$ and $\int_\Omega f\Psi$.
Assuming $\tilde{d}$ vanishing moments, rhs approximation based on

$$|\int_\Omega f\psi_\lambda| \leq \|\psi_\lambda\|_{L_2(\Omega)} \inf_{p\in P_{\tilde{d}-1}} \|f-p\|_{L_2(\operatorname{supp}\psi_\lambda)}.$$

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Similar arg. shows that **stiffness** is 'near-sparse'. Restricting it to fixed 'band' gives right complexity, but not suff. accuracy.
$\mathbf{u} \in \mathcal{A}^s$ means that **vector** is 'near-sparse'. One has $\|\mathbf{u}_i\|_{\mathcal{A}^s} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}$.
Approximate jth column of stiffness with accuracy proportional to $|(\mathbf{u}_i)_j|$.

Realizes cost condition. Quantitatively expensive.

An alternative residual evaluation

Ex. $\begin{cases} -u'' + u^3 = f \text{ on } (0,1), \\ u(0) = u(1) = 0. \end{cases}$ Piecew. pol. wav. basis Ψ for $H_0^1(0,1)$.

$$\mathbf{F}(\mathbf{u}_i) = \Big[\int_0^1 u_i'\psi_\lambda' + (u_i^3 - f)\psi_\lambda \, dx\Big]_{\lambda\in\nabla} = \Big[\int_0^1 \underbrace{(-u_i'' + u_i^3 - f)}\psi_\lambda \, dx\Big]_{\lambda\in\nabla},$$

(where $u_i := \mathcal{F}\mathbf{u}_i$) **assuming** $\Psi \subset H^2(0,1)$.
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Tree constraint on sets $\Lambda_i \subset \nabla$: If $\lambda \in \Lambda_i$, and $|\lambda| > 0$, then $\operatorname{supp}\psi_\lambda \subseteq \cup\{\operatorname{supp}\psi_\mu \colon |\mu| = |\lambda| - 1,\ \mu \in \Lambda_i\}$. Affects $\mathcal{A}^s$, but only slightly.

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Then u_i, and thus $-u_i'' + u_i^3 - f$, is piecewise polynomial w.r.t. partition $\mathcal{T}_i$; and its repr. $\mathbf{c}$ w.r.t. a single-scale basis Φ_i can be found in lin. compl. (**wavelet-to-single scale transf**).

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Dropping from $\mathbf{F}(\mathbf{u}_i)$ all λ whose levels exceed the level of $\mathcal{T}_i|_{\operatorname{supp}\psi_\lambda}$ by a fixed constant $k = k(\delta)$ gives a relative error δ.
With $\Lambda_{i,\delta}$ the remaining set of indices, let $\Phi_{i,\delta}$ be a single-scale basis for $\operatorname{span}\Psi|_{\Lambda_{i,\delta}}$. Compute $\mathbf{F}(\mathbf{u}_i)|_{\Lambda_{i,\delta}}$ by computing $[\int_0^1 \Phi_i \Phi_{i,\delta} dx]\mathbf{c}$, followed by a **single-scale-to-wavelet transf.** Total cost $\eqsim \#\Lambda_{i,\delta} \eqsim \Lambda_i$.

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Cost condition satisfied. Quantitatively more efficient. Generalizes to **non-linear** problems. **Inconvenient** condition on wavelets (in $d > 1$ dims.)

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- $G : \mathscr{X} \supset \mathrm{dom}(G) \to \mathscr{Y}'$,
- G 2x cont. Fréchet diff. in neighb. of a sol u of $\boxed{G(u) = 0,}$
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Necess. $u = \mathrm{argmin}_{v \in \mathrm{dom}(G)} \frac{1}{2}\|G(v)\|^2_{\mathcal{Y}'}$, and so (E-L), $F(u) = 0$, where $F : \mathcal{X} \supset \mathrm{dom}(F) \to \mathcal{X}'$

$$F(u)(v) := \langle DG(u)(v), G(u) \rangle_{\mathcal{Y}'}.$$

Having Riesz basis $\Psi_{\mathcal{X}}$ for $\mathcal{X}$, **awgm** applies.

Problem when $\langle\,,\,\rangle_{\mathcal{Y}'}$ is not evaluable: Equip $\mathcal{Y}$ with Riesz basis $\Psi_{\mathcal{Y}}$, and $\mathcal{Y}'$ with **equiv.** norm $\|\mathcal{F}'f\|$ $(= \|f(\Psi_{\mathcal{Y}})\|)$. Then

$$F(u)(v) := DG(u)(v)(\Psi_{\mathcal{Y}})^\top G(u)(\Psi_{\mathcal{Y}}),$$

and so $\mathbf{F}(\cdot)$ $(= \mathcal{F}'_{\mathcal{X}} F \mathcal{F}_{\mathcal{X}}) = D\mathbf{G}(\cdot)^\top \mathbf{G}(\cdot)$, where $\mathbf{G} = \mathcal{F}'_{\mathcal{Y}} G \mathcal{F}_{\mathcal{X}}$.

$Rem.$ If, however, $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N$, then only those $\mathcal{Y}_i$ with a non-evaluable inner product have to be equipped with Riesz bases.
This setting not covered by approach of first writing system in wavelet coordinates, and then forming (nonlinear) normal equations.

First order system least squares

Ex. $-\Delta u = f$ on Ω, $u = 0$ at $\partial\Omega$.

$$G\colon (u, \vec{p}) \mapsto (\operatorname{div}\vec{p} + f, \vec{p} - \mathbf{grad}\, u)\colon \underbrace{H_0^1(\Omega) \times H(\operatorname{div};\Omega)}_{\mathcal{X}} \to \underbrace{L_2(\Omega) \times L_2(\Omega)^d}_{\mathcal{Y}'}.$$

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where $F\colon \mathcal{X} \to \mathcal{X}'$ reads $F(u,\vec{p})(w,\vec{q}) = \langle DG(u,\vec{p})(v,\vec{q}), G(u,\vec{p})\rangle_{\mathcal{Y}'}$.

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awgm: Equip $H_0^1(\Omega)$ and $H(\operatorname{div};\Omega)$ (thus $\mathcal{X}$) with Riesz bases $\Psi^{H_0^1}$ and $\Psi^{H(\operatorname{div})}$

$$\mathbf{F}(\mathbf{u}_i,\mathbf{p}_i) = \begin{bmatrix} \langle \mathbf{grad}\,\Psi^{H_0^1}, \underbrace{\mathbf{grad}\,u_i - \vec{p}_i}\rangle_{L_2(\Omega)^d} \\ [\langle \operatorname{div}\Psi^{\operatorname{div}}, \underbrace{\operatorname{div}\vec{p}_i + f}\rangle_{L_2(\Omega)} + \langle \Psi^{\operatorname{div}}, \underbrace{\vec{p}_i - \mathbf{grad}\,u_i}\rangle_{L_2(\Omega)^d} \end{bmatrix}.$$

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Pros:

- efficient 'alternative residual evaluation' applies without additional smoothness requirements on wavelets.
- lower order (nonlinear) terms can be added (as with any least squares formulation).
- least squares minimalisation in $\mathcal{Y}' = L_2(\Omega)^{d+1}$ (convenient).

Numerics

L-shaped domain $\Omega \subset \mathbb{R}^2$. Bases for $H_0^1(\Omega)$ and $H(\mathrm{div};\Omega)$ consisting of cont. piecewise linears and lowest order RT-functions, resp.

$-\Delta u + N(u) = f$ on Ω, $u = 0$ at $\partial\Omega$, where $N(u) = u^3$ or $N(u) = \sin u$. $f = 1$

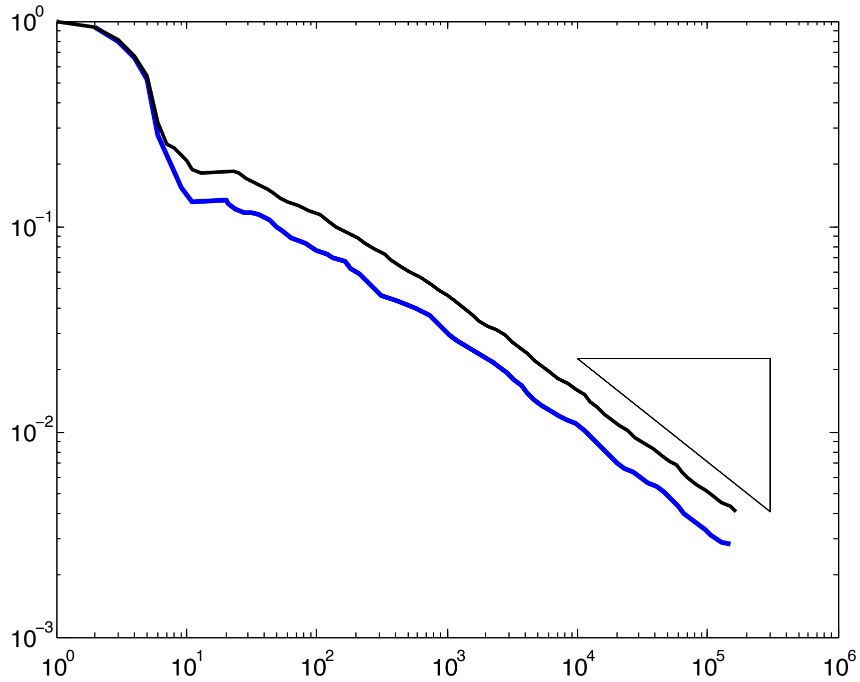

Figure 1: Norm of residual vs. number of wavelets in log-log scale, for $N(u) = u^3$ (black, upper curve) or $N(u) = \sin u$ (blue, lower curve). The hypotenuse of the triangle has a slope of $-\frac{1}{2}$.

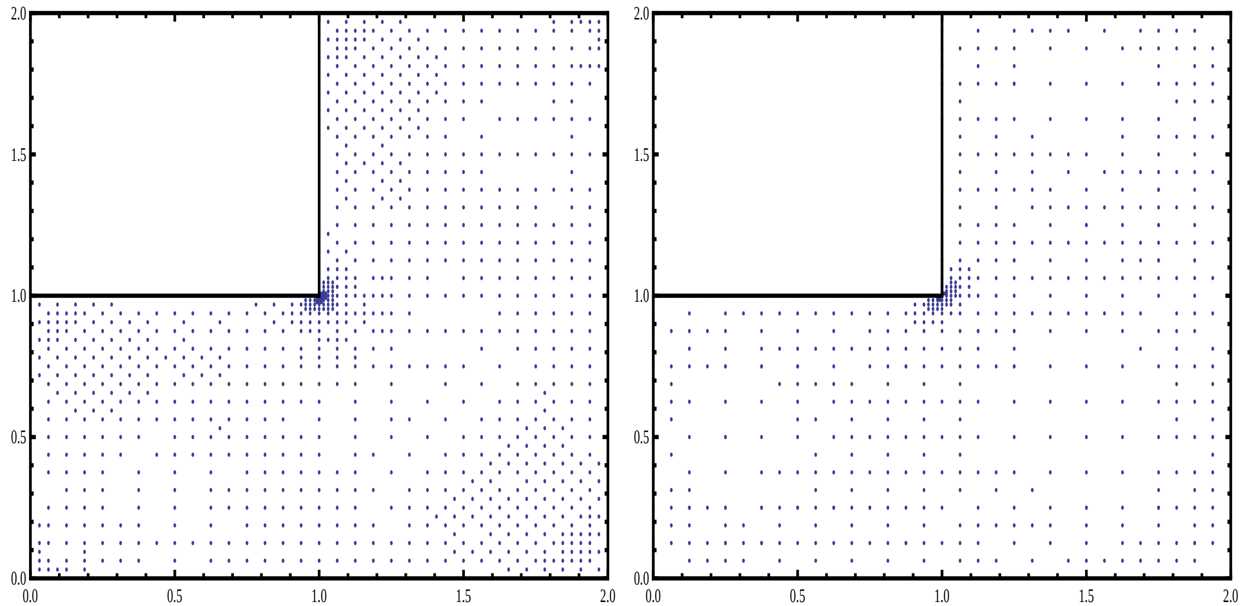

Figure 2: Centers of the supports of the wavelets in $H_0^1(\Omega)$ for the approximation of u (left, 930 wavelets), or the wavelets in $H(\mathrm{div};\Omega)$ for the approximation of $\vec{p}$ (right, 631 wavelets) produced by **awgm** after 39 iterations for $N(u) = u^3$.

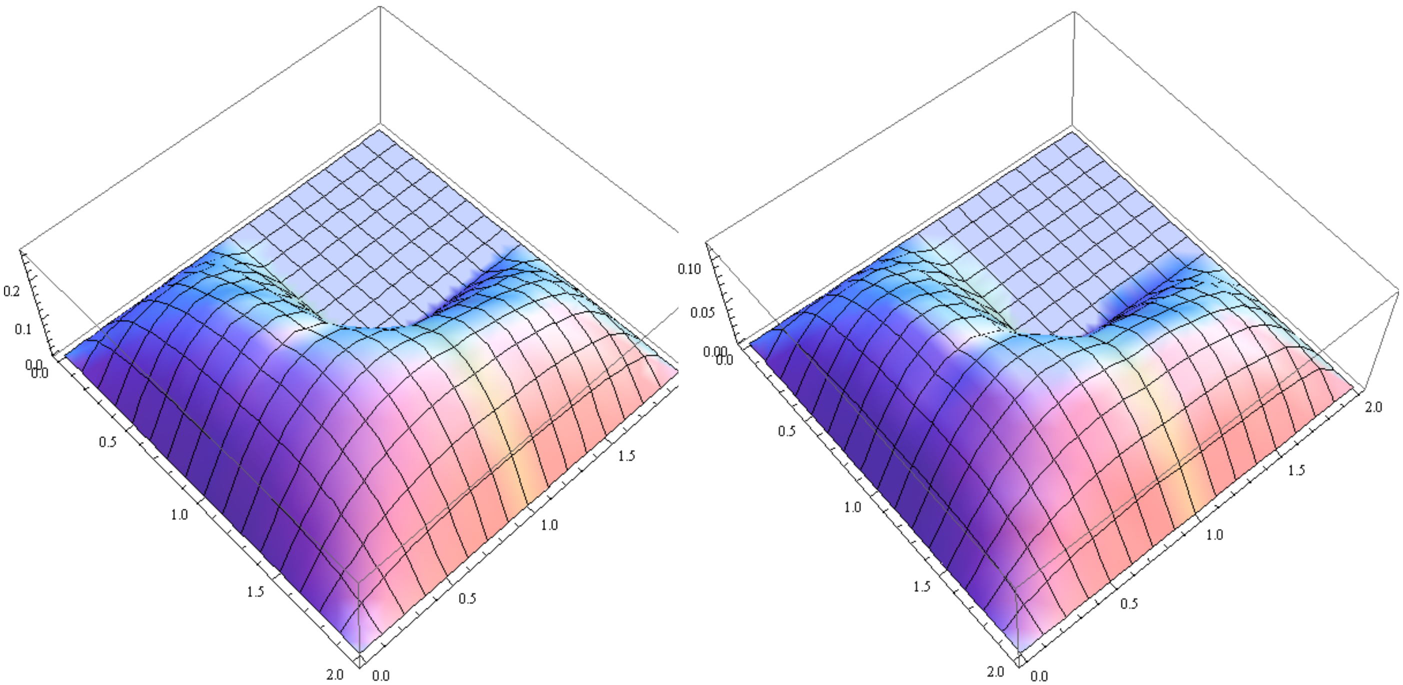

Figure 3: Approximate solutions for $N(u) = u^3$ (left) or $N(u) = \sin u$ (right), as a linear combination of approximately 200 wavelets. Note the difference in vertical scale in both pictures.

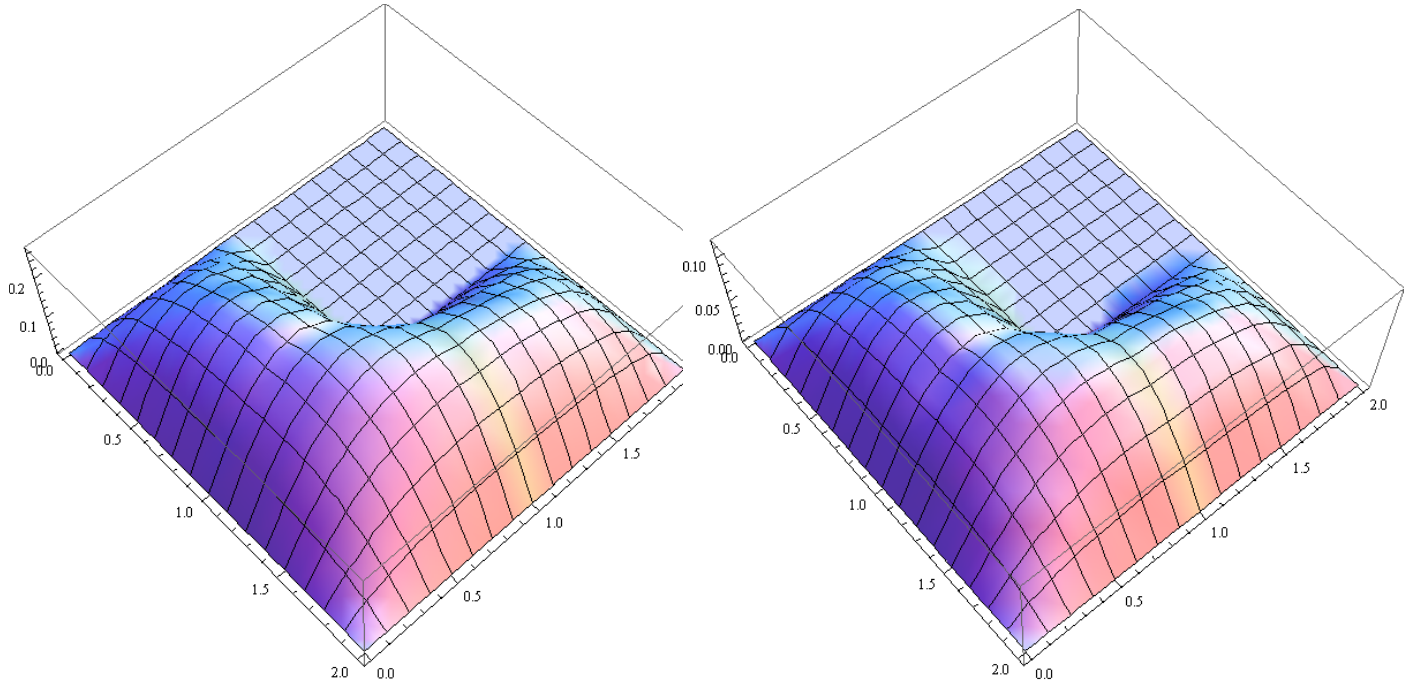

Figure 3: Approximate solutions for $N(u) = u^3$ (left) or $N(u) = \sin u$ (right), as a linear combination of approximately 200 wavelets. Note the difference in vertical scale in both pictures.

Cons current first order system formulation:

- Requires wavelet basis for $\mathbf{H}(\mathrm{div}; \mathbf{\Omega})$. Realized in two dims only.
- For $-\triangle u + N(u) = f$, needed $N : H_0^1(\Omega) \to L_2(\Omega)$ and $f \in L_2(\Omega)$.
- Not a 'canonical' approach.

First order system least squares (revisited)

Ex. $-\Delta u = f$ on Ω, $u = 0$ at $\partial\Omega$.

$$G\colon (u,\vec{p}) \mapsto (f - \mathbf{grad}'\,\vec{p}, \vec{p} - \mathbf{grad}\,u)\colon \underbrace{H_0^1(\Omega) \times \mathbf{L_2(\Omega)^d}}_{\mathcal{X}} \to \underbrace{\mathbf{H^{-1}(\Omega)} \times L_2(\Omega)^d}_{\mathcal{Y}'}.$$

$DG(u,\vec{p}) = DG \in \mathcal{L}\mathrm{is}(\mathcal{X},\mathcal{Y}')$. Least-squares minimalisation, E-L $\rightsquigarrow F(u,\vec{p}) = 0$,

where $F\colon \mathcal{X} \to \mathcal{X}'$ reads $F(u,\vec{p})(w,\vec{q}) = \langle DG(u,\vec{p})(v,\vec{q}), G(u,\vec{p})\rangle_{\mathcal{Y}'}$.

Equip $H_0^1(\Omega)$ with Riesz basis $\Psi^{H_0^1}$, and repl. $\|\cdot\|_{H^{-1}(\Omega)}$ by $\|\mathcal{F}'_{H_0^1}\cdot\|$.

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Equip $H_0^1(\Omega)$ with Riesz basis $\Psi^{H_0^1}$, and repl. $\|\cdot\|_{H^{-1}(\Omega)}$ by $\|\mathcal{F}'_{H_0^1} \cdot\|$.

$$\begin{aligned}
F(u_i,\vec{p}_i)(v,\vec{q}) &= \langle \vec{q}, \mathbf{grad}\,\Psi^{H_0^1}\rangle_{L_2(\Omega)^d}\big[\langle \Psi^{H_0^1}, f\rangle_{L_2(\Omega)} - \langle \mathbf{grad}\,\Psi^{H_0^1}, \vec{p}_i\rangle_{L_2(\Omega)^d}\big] \\
&\quad + \langle \vec{q} - \mathbf{grad}\, v, \vec{p}_i - \mathbf{grad}\, u_i\rangle_{L_2(\Omega)^d} \\
&= \langle \vec{q}, \mathbf{grad}\,\Psi^{H_0^1}\rangle_{L_2(\Omega)^d}\big[\langle \Psi^{H_0^1}, f + \mathrm{div}\,\vec{p}_i\rangle_{L_2(\Omega)^d}\big] \\
&\quad + \langle \vec{q} - \mathbf{grad}\, v_i, \vec{p}_i - \mathbf{grad}\, u_i\rangle_{L_2(\Omega)^d}
\end{aligned}$$

if $\vec{p}_i \in H(\mathrm{div};\Omega)$.

awgm: Equip $H_0^1(\Omega)$, $L_2(\Omega)^d$ with Riesz bases $\Psi^{H_0^1}$, $\Psi^{L_2^d}$, where $\boldsymbol{\Psi^{L_2^d}} \subset \boldsymbol{H(\mathbf{div}\,;\Omega)}$. Then

$$\mathbf{F}(\mathbf{u}_i, \mathbf{p}_i) =$$

$$\begin{bmatrix} \langle \mathbf{grad}\, \Psi^{H_0^1}, \underbrace{\mathbf{grad}\, u_i - \vec{p}_i} \rangle_{L_2(\Omega)^d} \\ \langle \Psi^{L_2^n}, \mathbf{grad}\, \Psi^{H_0^1} \rangle_{L_2(\Omega)^d} \langle \Psi^{H_0^1}, \underbrace{\mathrm{div}\, \vec{p}_i + f} \rangle_{L_2(\Omega)^d} + \langle \Psi^{L_2^n}, \underbrace{\vec{p}_i - \mathbf{grad}\, u_i} \rangle_{L_2(\Omega)^d} \end{bmatrix}.$$

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Pros:

- efficient 'alternative residual evaluation' applies under mild condition.
- wavelet bases available in general settings (no basis for $H(\mathrm{div};\Omega)$ required)
- lower order (nonlinear) terms $N : H_0^1(\Omega) \to H^{-1}(\Omega)$ can be added. $f \in H^{-1}(\Omega)$ allowed.
- 'canonical' approach: Well-posedness of first order formulation follows from that of second order formulation. Applies equally well to time evolution problems.

Parabolic problems

$\Omega \subset I\!R^d$, $\mathrm{I} = (0, T)$.

$$\begin{cases} \frac{\partial u}{\partial t} - \triangle u = g & \text{on } \mathrm{I} \times \Omega, \\ u = 0 & \text{on } \mathrm{I} \times \partial\Omega, \\ u(0, \cdot) = 0 & \text{on } \Omega. \end{cases}$$

- $-\triangle$ can be read as semi-linear elliptic operator.
- other (inhom) initial or boundary conditions are allowed.

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- other (inhom) initial or boundary conditions are allowed.

Standard appr.: Approx. $\frac{\partial u}{\partial t}(t, \cdot)$ by, say $\frac{u(t,\cdot) - u(t-h,\cdot)}{h}$, and solve seq. of elliptic problems for $0 < t_1 < t_2 < \cdots < t_M = T$

$$\begin{cases} -\triangle u(t_i, \cdot) - (t_i - t_{i-1})^{-1} u(t_i, \cdot) = (t_i - t_{i-1})^{-1} u(t_{i-1}, \cdot) + g(t_i, \cdot) & \text{on } \Omega \\ u(t_i, \cdot) = 0 & \text{on } \partial\Omega \end{cases}$$

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- How to distribute optimally 'grid points' over space and time?
- Even if you can, approximation not effective for singularities that are local in space and time.
- Inherently sequential.
- When the whole time evolution is needed, as with problems of optimal control or in visualizations, huge amount of storage.

Space-time variational formulation

$$(Gu)(v) := \int_{\mathrm{I}}\int_\Omega \frac{\partial u}{\partial t} v + \mathbf{grad}\, u \cdot \mathbf{grad}\, v \, dx\, dt - \int_{\mathrm{I}}\int_\Omega gv\, dx\, dt = 0.$$

$$DG(u) = DG \in \mathcal{L}\mathrm{is}\Big(\underbrace{L_2(\mathrm{I}; H_0^1(\Omega)) \cap H^1_{0,\{0\}}(\mathrm{I}; H^{-1}(\Omega))}_{\mathcal{X} :=}, \underbrace{L_2(\mathrm{I}; H_0^1(\Omega))}_{\mathcal{Y} :=}{}'\Big).$$

After selecting Riesz $\Psi^{\mathcal{X}}$, $\Psi^{\mathcal{Y}}$ for $\mathcal{X}$, $\mathcal{Y}$, apply **awgm** to $D\mathbf{G}^\top \mathbf{G}(\mathbf{u}) = 0$.

(even better first to write it as a well-posed first order system)

Tensor product bases

Let $\Theta^{\mathcal{X}}$, $\Theta^{\mathcal{Y}}$, and $\Sigma^{\mathcal{X}}$, $\Sigma^{\mathcal{Y}}$ be collections of temporal or spatial functions such that, normalized in the corresponding norms,

$\Theta^{\mathcal{X}}$	is a Riesz basis for	$L_2(I)$	**and** for	$H^1_{0,\{0\}}(I)$,
$\Theta^{\mathcal{Y}}$	"	$L_2(I)$,		
$\Sigma^{\mathcal{X}}$	"	$H^1_0(\Omega)$	"	$H^{-1}(\Omega)$,
$\Sigma^{\mathcal{Y}}$	"	$H^1_0(\Omega)$.		

Then, normalized,

$\Theta^{\mathcal{X}} \otimes \Sigma^{\mathcal{X}}$	is a Riesz basis for	$L_2(I; H^1_0(\Omega))$, $H^1_{0,\{0\}}(I; H^{-1}(\Omega))$, and so for $\mathcal{X}$,
$\Theta^{\mathcal{Y}} \otimes \Sigma^{\mathcal{Y}}$	"	$\mathcal{Y}$.

Best possible rates

If $\Theta^{\mathcal{X}}$ and $\Sigma^{\mathcal{X}}$ are of orders p_t and p_x, then best possible approximation rate in $\mathcal{X}$ is $\min(p_t - 1, \frac{p_x - 1}{d})$.
Rate requires boundedness of certain mixed derivatives of sol in L_p for some $p < 2$ ($p = 2$ required for sparse-grids). Approx. classes can be characterized as tensor products of Besov spaces.

For $p_t - 1 \geq \frac{p_x - 1}{d}$, best rate is **equal** to best possible approx. rate in $H_0^1(\Omega)$ using $\Sigma^{\mathcal{X}}$.
So thanks to tensor product basis, **no** penalty because of additional time dimension.

First test on an ODE

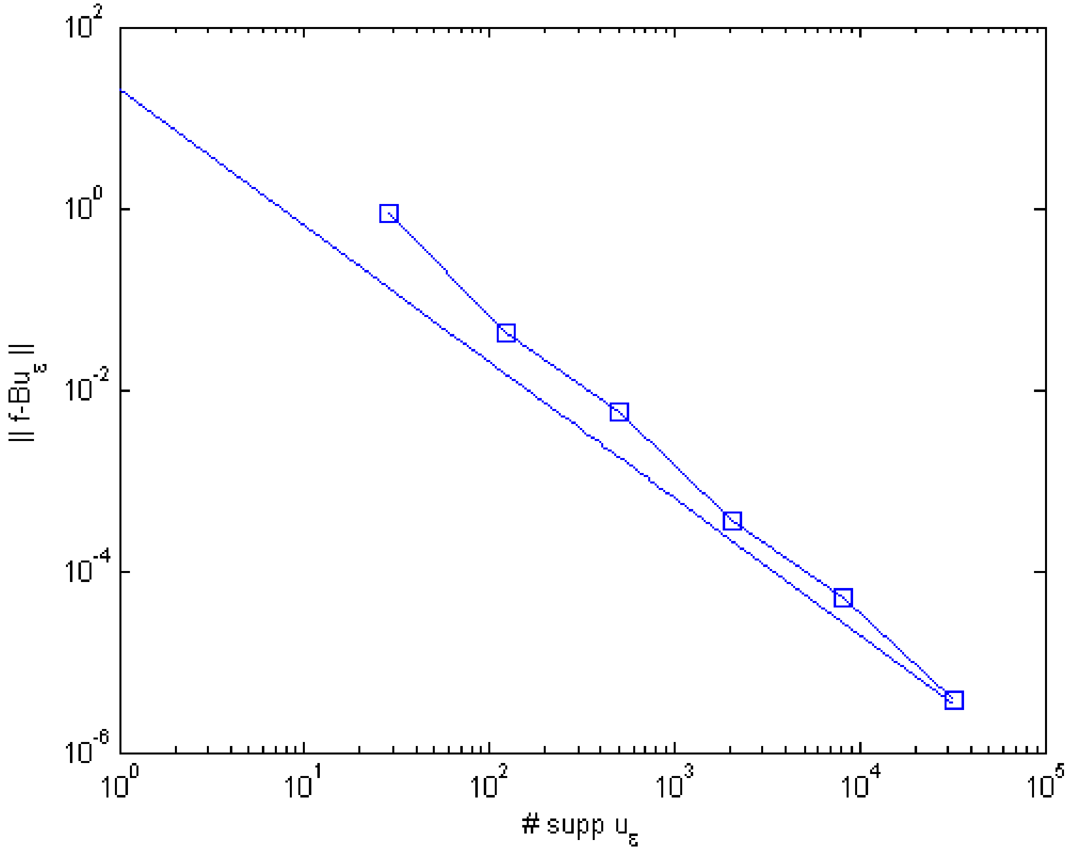
$$\begin{cases} \frac{du(t)}{dt} + \nu u(t) = g(t) & (t \in \mathrm{I}), \\ u(0) = u_0, \end{cases}$$

$$(Gu)(v) := \int_{\mathrm{I}} -u(t)\frac{dv(t)}{dt} + \nu u(t)v(t)dt - \int_{\mathrm{I}} g(t)v(t)dt - u_0 v(0) = 0.$$

Prop. *With* $\mathcal{X} := L_2(\mathrm{I})$ *and* $\mathcal{Y}(\nu) := H^1_{0,\{T\}}(\mathrm{I})$, *equipped with* $\|\cdot\|_{\mathcal{Y}(\nu)} := \sqrt{\nu^2\|\cdot\|^2_{L_2(\mathrm{I})} + |\cdot|^2_{H^1(\mathrm{I})}}$, *the operator* $DG \in \mathcal{L}\mathrm{is}(\mathcal{X}, \mathcal{Y}(\nu)')$ *and* $\|DG\| \leq \sqrt{2}$, $\|DG^{-1}\| \leq \sqrt{2}$.

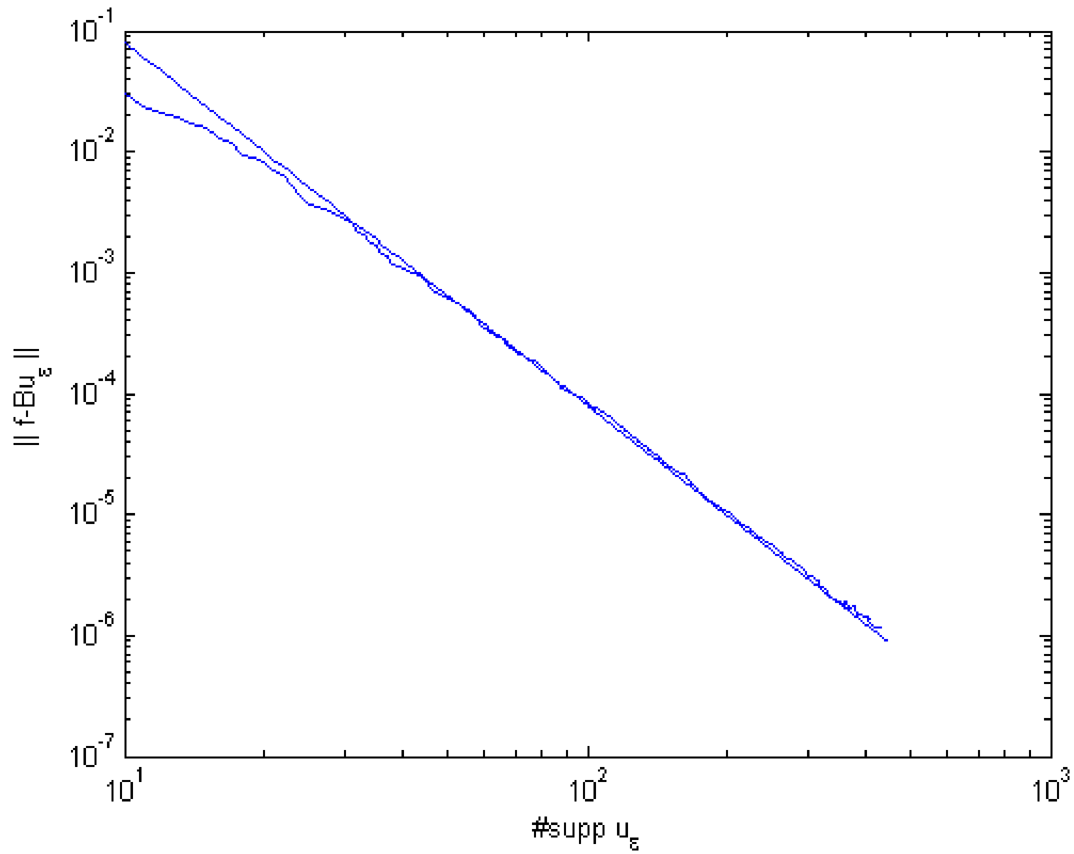
Num. results for $\nu = 1$, $g = 1$ on $(0, \frac{1}{3})$, $g = 2$ on $(\frac{1}{3}, 1)$.

Uniform, non-adaptive refinements, i.e. collect all wavelets up to some level.



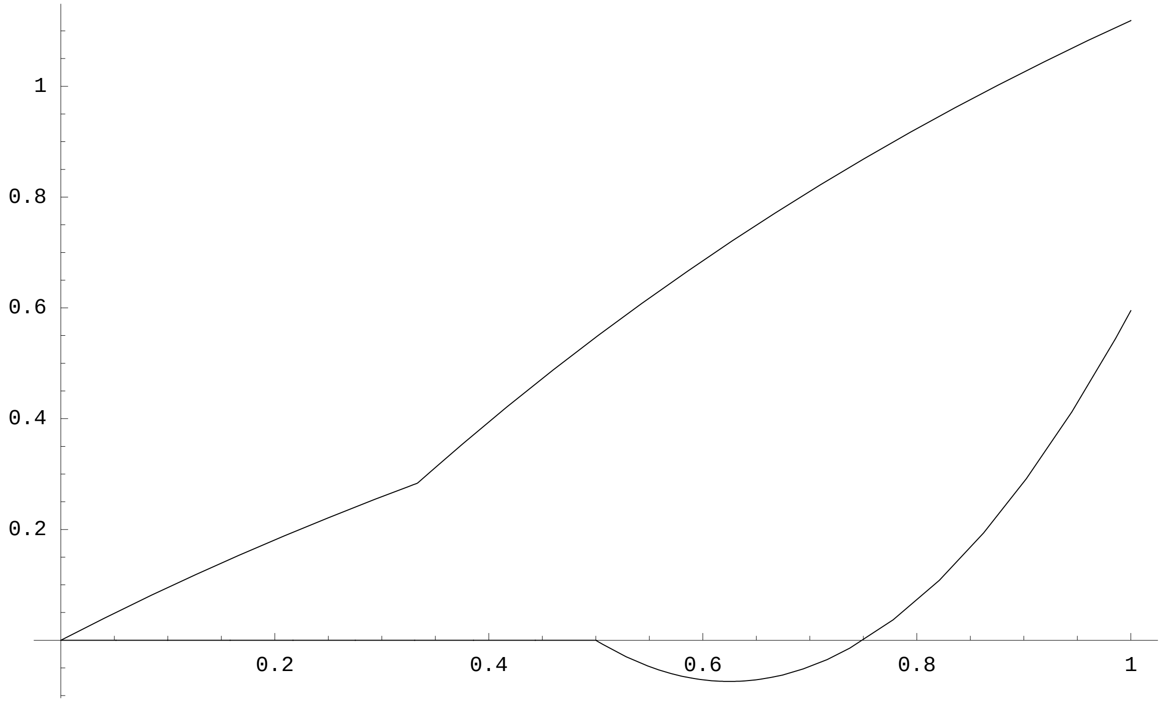
Rate $= 1.5$

Adaptive refinements, i.e. **awgm**

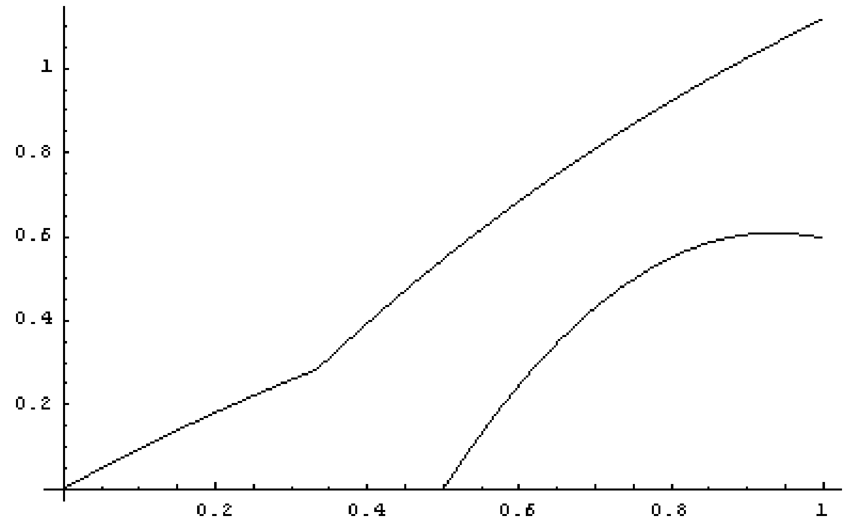


Rate $= 3$

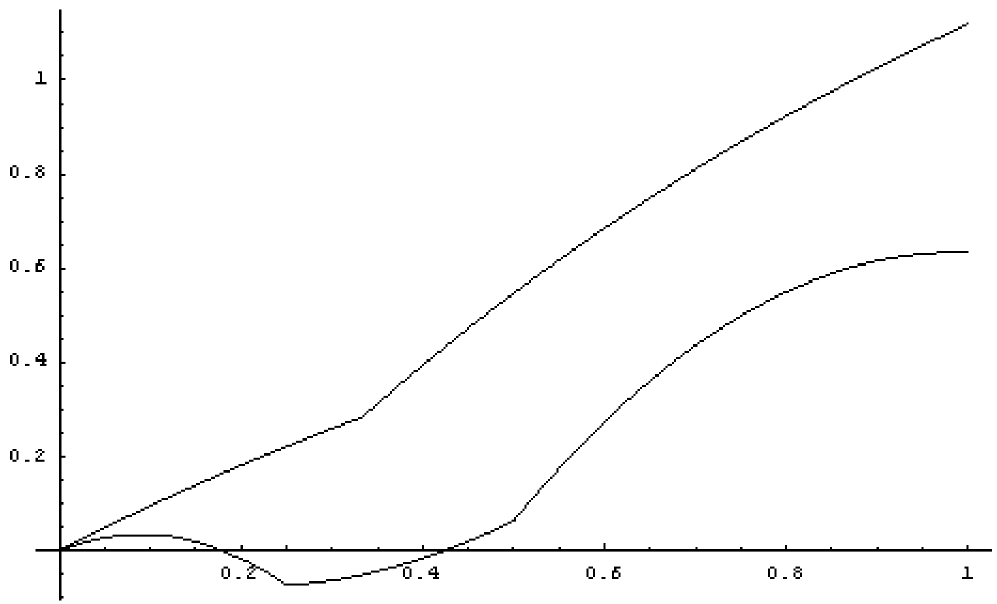
Some approximations



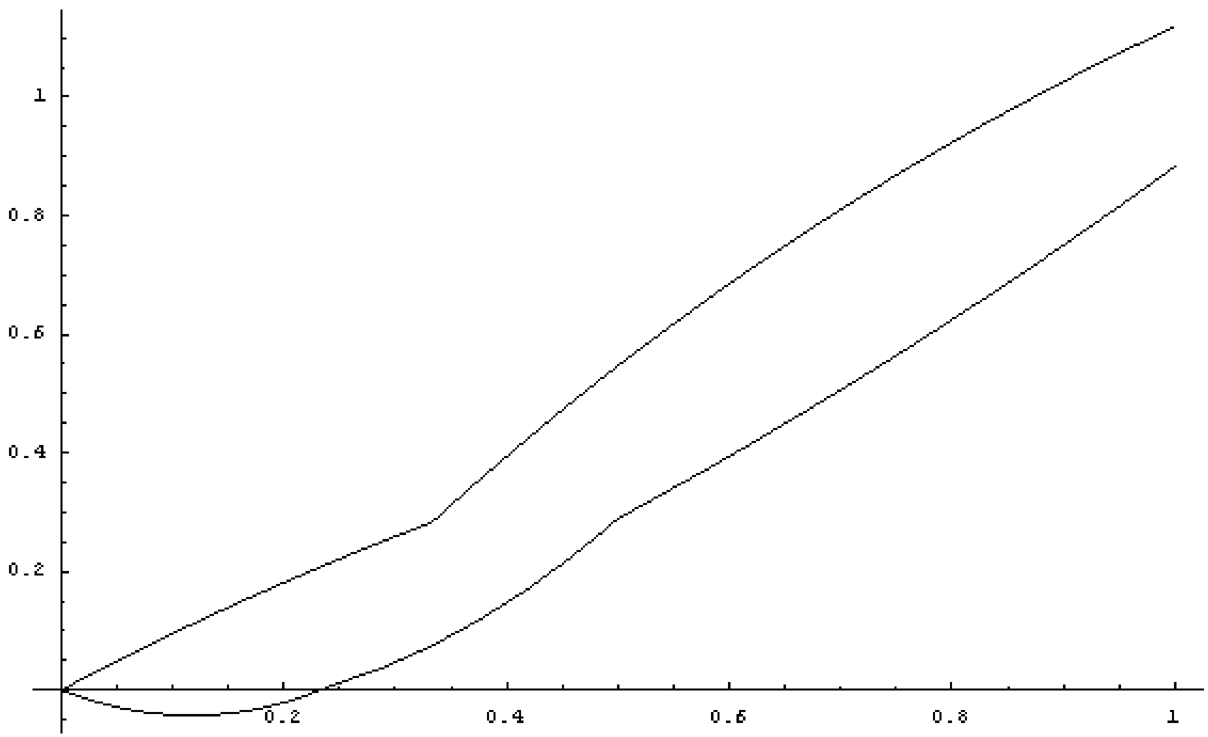
1 wavelet



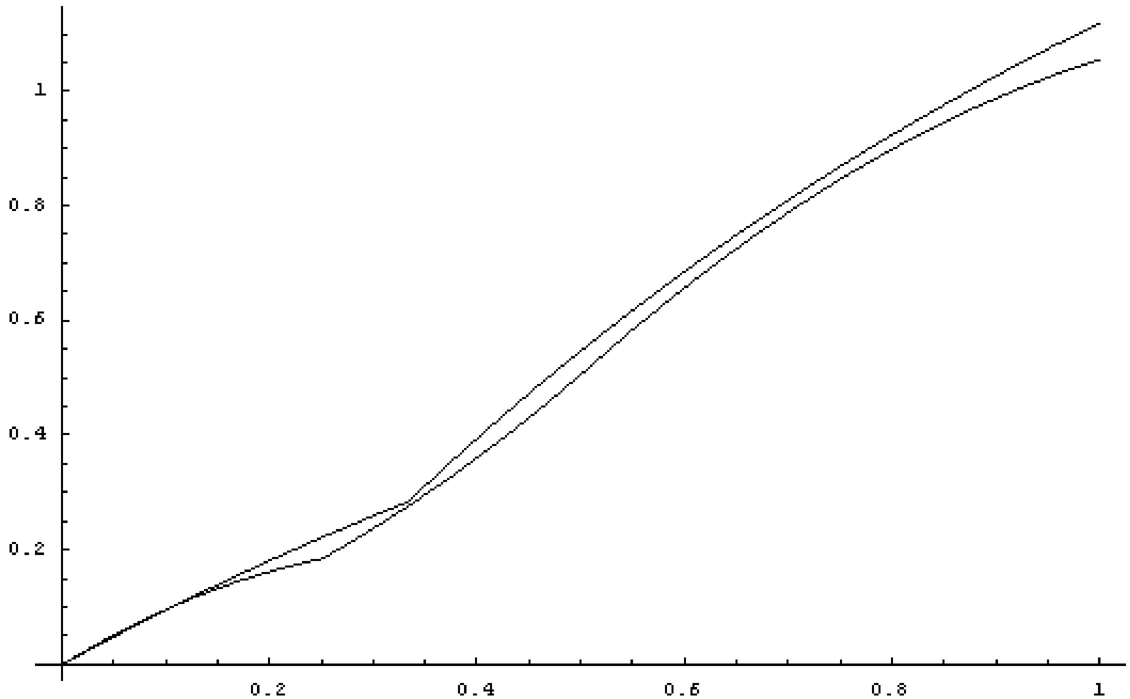
2 wavelets



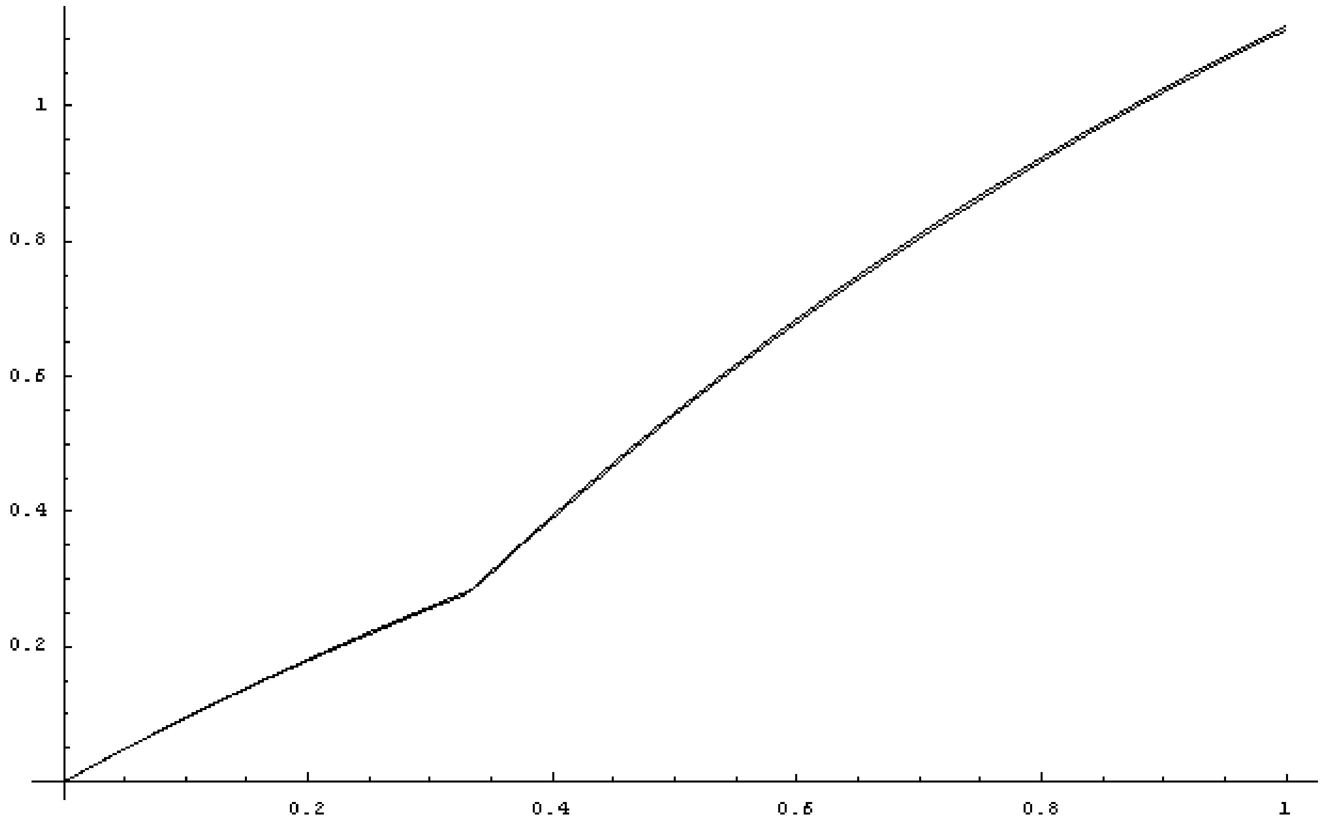
3 wavelets



4 wavelets



5 wavelets (all scaling functions are now in)



15 wavelets

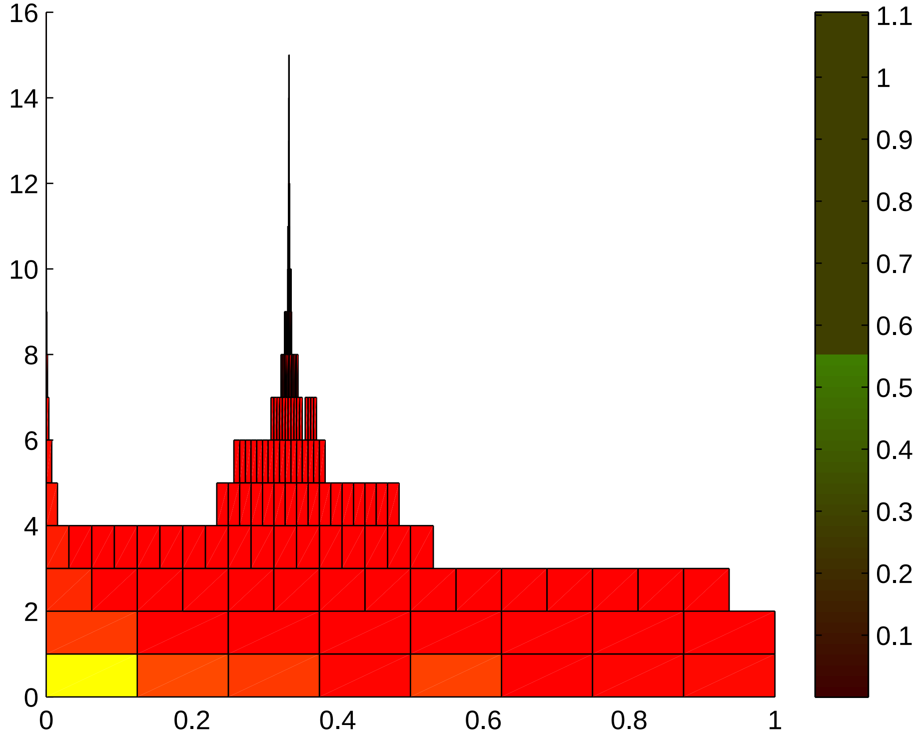


Figure 4: For $u_0 = 1$ and $\#\mathbf{u}_i = 202$, the non-zero coefficients of $\mathbf{u}_i$.

Numerical results heat eqn

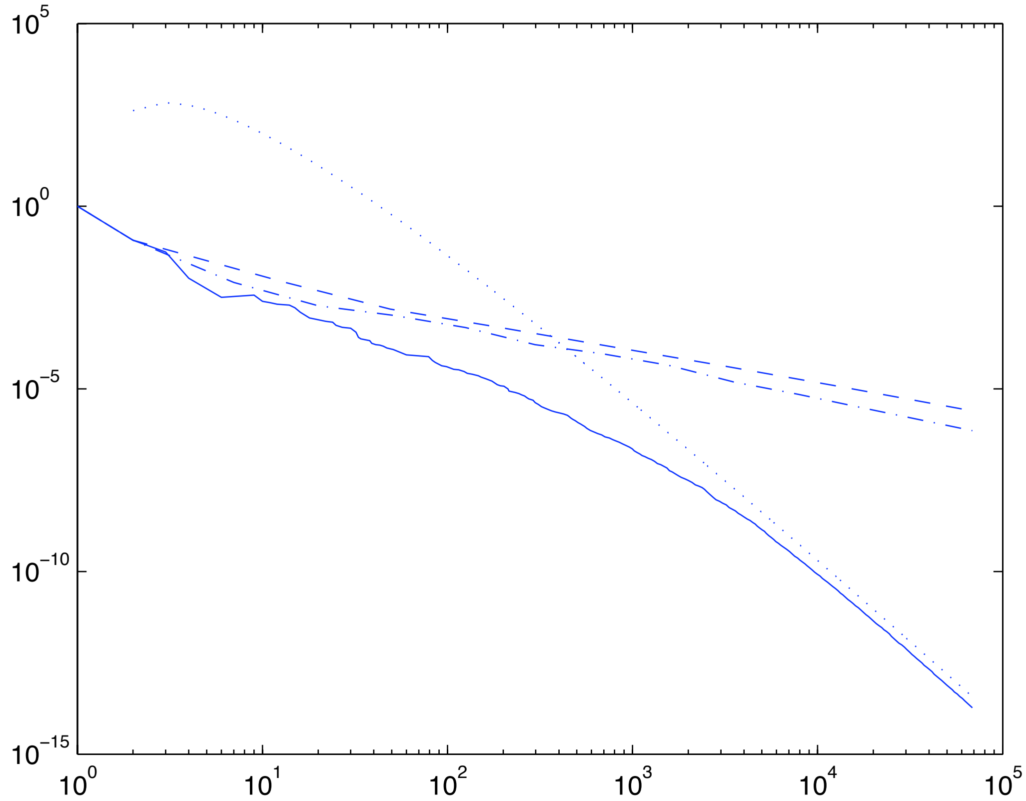


Figure 5: Heat eqn. in $d = 1$ spatial dimension, right-hand side $g = 1$ and initial condition $u_0 = 0$. $\|\mathbf{G}(\mathbf{u}_i)\|/\|\mathbf{G}(0)\|$ vs. $N = \#\mathrm{supp}\,\mathbf{u}_i$ for the **awgm** (solid), full-grid (dashed) and sparse-grid method (dashed-dotted). The dotted line is a multiple of $N^{-5}(\log N)^{5\frac{1}{2}}$.

Numerical results heat eqn

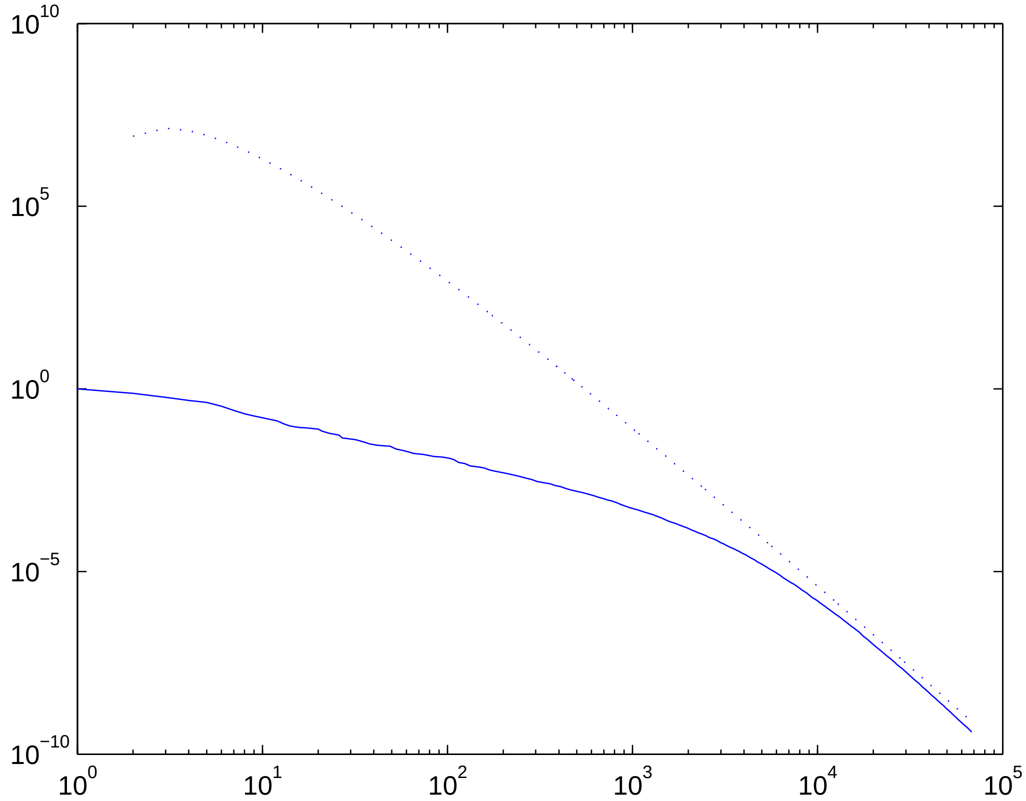


Figure 6: Heat eqn. in $d = 1$ spatial dimension, right-hand side $g = 1$ and initial condition $u_0 = 1$. $\|\mathbf{G}(\mathbf{u}_i)\|/\|\mathbf{G}(0)\|$ vs. $N = \#\mathrm{supp}\,\mathbf{u}_i$ for the **awgm** (solid). The dotted line is a multiple of $N^{-5}(\log N)^{5\frac{1}{2}}$.

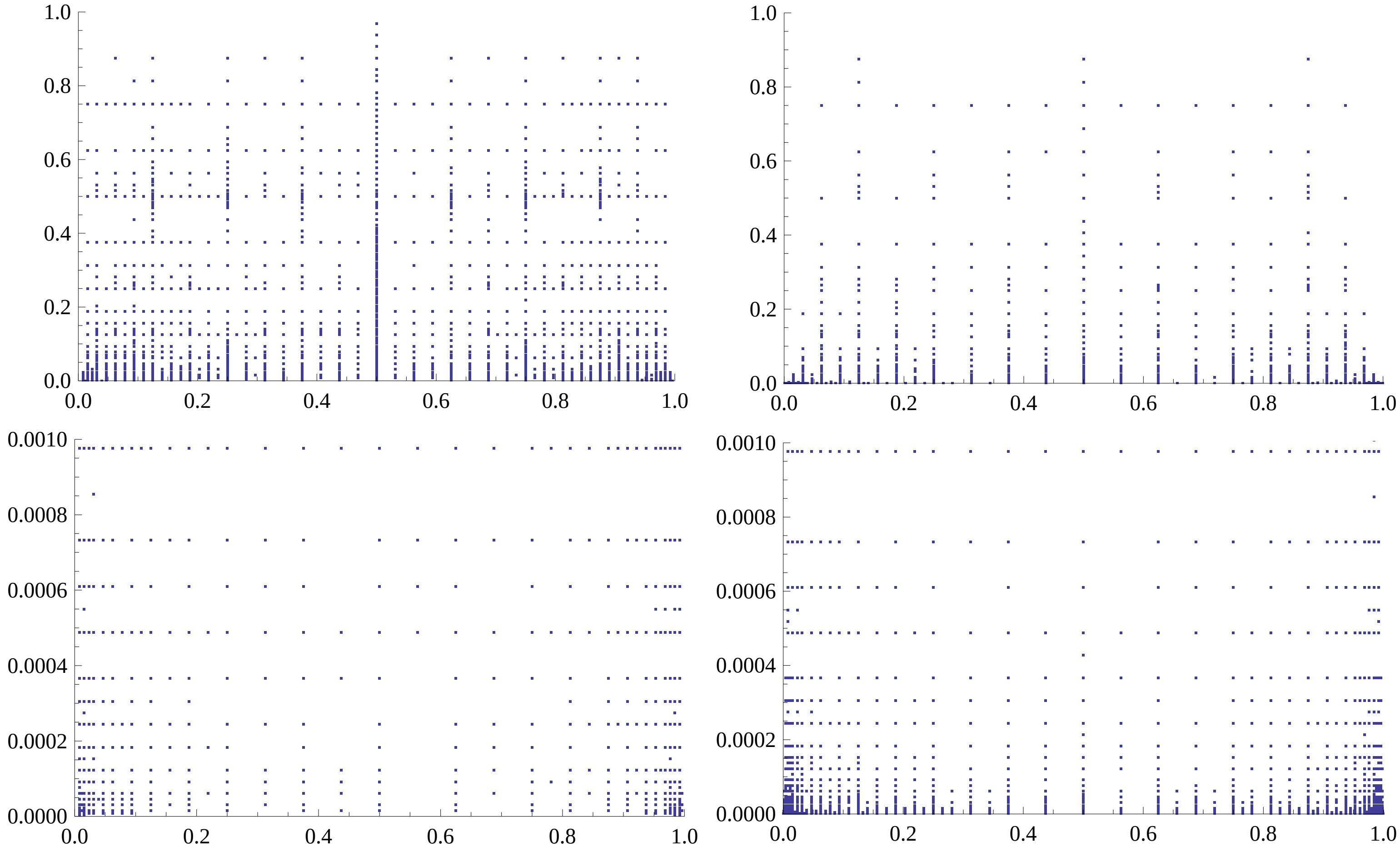


Figure 7: Heat eqn. in $d = 1$ spatial dimension and right-hand side $g = 1$. Centers of the supports of the wavelets selected by the **awgm**. Left $u_0 = 0$ and $\#\mathbf{u}_i = 13420$. Right $u_0 = 1$ and $\#\mathbf{u}_i = 13917$. A zoom in near $t = 0$ is given at the bottom row.

(N)SE

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} p = \mathbf{f} & \text{on } I \times \Omega, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u} = g & \text{on } I \times \Omega, \\ \mathbf{u} = 0 & \text{on } I \times \partial\Omega, \\ \mathbf{u}(0, \cdot) = 0 & \text{on } \Omega, \\ \int_\Omega p \, d\mathbf{x} = 0. & \end{cases} \quad (1)$$

Can be reduced to parabolic for velocities, but then arising spaces will be spaces of divergence-free functions. We enforce incompressibility constraint via Lagrange multiplier. Saddle point form.

(N)SE

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \mathbf{\Delta}_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} p = \mathbf{f} & \text{on } I \times \Omega, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u} = g & \text{on } I \times \Omega, \\ \mathbf{u} = 0 & \text{on } I \times \partial\Omega, \\ \mathbf{u}(0, \cdot) = 0 & \text{on } \Omega, \\ \int_\Omega p \, d\mathbf{x} = 0. \end{cases} \tag{1}$$

Can be reduced to parabolic for velocities, but then arising spaces will be spaces of divergence-free functions. We enforce incompressibility constraint via Lagrange multiplier. Saddle point form.

Space-time variational form: With

$$\begin{cases} c(\mathbf{u}, \mathbf{v}) & := \int_I \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} + \nu \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathbf{v} \, d\mathbf{x} dt, \\ d(p, \mathbf{v}) & := -\int_I \int_\Omega p \operatorname{div} \mathbf{v} \, d\mathbf{x} dt, \\ \mathbf{f}(\mathbf{v}) & := \int_I \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} dt, \\ g(q) & := \int_I \int_\Omega g \, q \, d\mathbf{x} dt, \end{cases} \tag{2}$$

find $(\mathbf{u}, p)$ in some suitable space, such that

$$G(\mathbf{u},p)(\mathbf{v},q) := c(\mathbf{u},\mathbf{v}) + d(p,\mathbf{v}) + d(q,\mathbf{u}) - \mathbf{f}(\mathbf{v}) + g(q) = 0$$

for all $(\mathbf{v},q)$ from another suitable space.

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for all $(\mathbf{v},q)$ from another suitable space. For $\delta \in \{0,T\}$,

$$\begin{aligned}
\breve{H}^s_{0,\{\delta\}}(I) &:= [L_2(I), H^1_{0,\{\delta\}}(I)]_s,\\
\hat{H}^s(\Omega) &:= [L_2(\Omega), H^2(\Omega)\cap H^1_0(\Omega)]_{\frac{s}{2}},\\
\bar{H}^s(\Omega) &:= [(H^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R}]_{\frac{s+1}{2}},\\
\mathscr{U}^s_\delta &:= L_2(I;\hat{H}^{2s}(\Omega)^n)\cap \breve{H}^s_{0,\{\delta\}}(I;L_2(\Omega)^n),\\
\mathscr{P}^s_\delta &:= \big(L_2(I;\bar{H}^{2s-1}(\Omega)')\cap \breve{H}^{1-s}_{0,\{\delta\}}(I;\bar{H}^1(\Omega)')\big)'.
\end{aligned}$$

Thm. *For $\Omega \subset \mathbb{R}^d$ a bounded Lipschitz domain, and $s \in (\frac{1}{4},\frac{3}{4})$, it holds that*

$$DG \in \mathcal{L}\mathrm{is}(\mathscr{U}^s_0 \times \mathscr{P}^s_T, (\mathscr{U}^{1-s}_T \times \mathscr{P}^{1-s}_0)').$$

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All arising spaces can be ‘conveniently’ equipped with wavelet Riesz bases, and **awgm** applies (preferably to reformulation as first order system)

Generalizes to NSE for $d=2$; for $d=3$ we need ‘s’ $>\frac{3}{4}$ which requires more smooth or convex domains, and C^1-wavelets.

Proof of Thm.

Recall saddle-point structure $DG(\mathbf{u},p)(\mathbf{v},q) := c(\mathbf{u},\mathbf{v}) + d(p,\mathbf{v}) + d(q,\mathbf{u})$. Boundedness is easy.

The right-inverse div^+ of div constructed in [Bog79] satisfies **both** $\operatorname{div}^+ \in \mathcal{L}(\bar{H}^{-1}(\Omega), L_2(\Omega)^n)$ and, for $s \in [0, \frac{3}{4})$, $\operatorname{div}^+ \in \mathcal{L}(\bar{H}^{2s-1}(\Omega), \hat{H}^{2s}(\Omega)^n)$, and so $I \otimes \operatorname{div}^+ \in \mathcal{L}((\mathscr{P}_0^{1-s})', \mathscr{U}_0^s)$. This implies that for $s \in [0, \frac{3}{4})$, $I \otimes \operatorname{div} \in \mathcal{L}(\mathscr{U}_0^s, (\mathscr{P}_0^{1-s})')$ is surjective, i.e.,

$$\inf_{0 \neq q \in \mathscr{P}_0^{1-s}} \sup_{0 \neq \mathbf{u} \in \mathscr{U}_0^s} \frac{d(\mathbf{u},q)}{\|\mathbf{u}\|_{\mathscr{U}_0^s} \|q\|_{\mathscr{P}_0^{1-s}}} > 0,$$

and analogously, for $s \in (\frac{1}{4}, 1]$,

$$\inf_{0 \neq p \in \mathscr{P}_T^s} \sup_{0 \neq \mathbf{v} \in \mathscr{U}_T^{1-s}} \frac{d(\mathbf{v},p)}{\|\mathbf{v}\|_{\mathscr{U}_T^{1-s}} \|p\|_{\mathscr{P}_T^s}} > 0.$$

Remains to show that $(C\mathbf{u})(\mathbf{v}) := c(\mathbf{u},\mathbf{v})$ boundedly inv. between $\{\mathbf{u} \in \mathscr{U}_0^s \colon d(\mathscr{P}_0^{1-s}, \mathbf{u}) = 0\}$ and $\big(\{\mathbf{v} \in \mathscr{U}_T^{1-s} \colon d(\mathscr{P}_T^s, \mathbf{v}) = 0\}\big)'$.

Again the existence of div^+ as constructed in [Bog79] shows that for $(\varsigma, \delta) \in \{(s,0),(1-s,T)\}$

$$\{\mathbf{w} \in \mathscr{U}_\delta^\varsigma \colon d(\mathscr{P}_\delta^{1-\varsigma}, \mathbf{w}) = 0\} \\ \simeq L_2(I; \hat{H}^{2\varsigma}(\operatorname{div} 0; \Omega)) \cap \breve{H}^\varsigma_{0,\{\delta\}}(I; \hat{H}^0(\operatorname{div} 0; \Omega)) =: \mathscr{U}_\delta^\varsigma(\operatorname{div} 0),$$

i.e. the order of interpolation and taking divergence-free parts can be reversed.

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i.e. the order of interpolation and taking divergence-free parts can be reversed.

With $(A\mathbf{u})(\mathbf{v}) := \nu \int_\Omega \nabla \mathbf{u} \colon \nabla \mathbf{v}\, d\mathbf{x}$ on $\hat{H}^1(\operatorname{div} 0; \Omega) \times \hat{H}^1(\operatorname{div} 0; \Omega)$, **elliptic regularity** shows that for $\varsigma \in [0, \frac{3}{4})$, $\hat{H}^{2\varsigma}(\operatorname{div} 0; \Omega) \simeq [\hat{H}^0(\operatorname{div} 0; \Omega), D(A)]_\varsigma$ and so

$$\mathscr{U}_\delta^\varsigma(\operatorname{div} 0) \simeq L_2(I; [\hat{H}^0(\operatorname{div} 0; \Omega), D(A)]_\varsigma) \cap \breve{H}^\varsigma_{0,\{\delta\}}(I; \hat{H}^0(\operatorname{div} 0; \Omega)) =: \tilde{\mathscr{U}}_\delta^\varsigma(\operatorname{div} 0)$$

Finally,

$$C \in \mathcal{L}\mathrm{is}(\tilde{\mathscr{U}}_0^\varsigma(\operatorname{div} 0), (\tilde{\mathscr{U}}_T^{1-\varsigma}(\operatorname{div} 0))') \quad (\varsigma \in [0,1]),$$

follows from interpolation and this result for $\varsigma \in \{0,1\}$, which results are known as **maximal regularity** of evolution equations.

Summary

- Adaptive wavelet methods solve well-posed operator equations at optimal rates, in linear comput. complexity

- Quantitative improvements by writing the problem as a first order system

- Promising applications for solving time evolution problems

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Thanks for your attention/patience!

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