

AN ADAPTIVE ALGORITHM FOR PDE PROBLEMS
WITH RANDOM DATA

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Background:

Alex Bespalov, Catherine Powell and David Silvester

Energy norm a posteriori error estimation for parametric operator equations,
SIAM J. Scientific Computing, **36**, A339–A363, 2014.

<http://dx.doi.org/10.1137/130916849>

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Software: Stochastic IFISS (S-IFISS) version 1.02, July 2015

<http://www.manchester.ac.uk/ifiss/s-ifiss1.0.tar.gz>

Motivating example (target model problem)

Find $u(\mathbf{x}, \mathbf{y})$ satisfying

$$\begin{aligned} -\nabla \cdot (a(\mathbf{x}, \mathbf{y}) \nabla u(\mathbf{x}, \mathbf{y})) &= f(\mathbf{x}) & \mathbf{x} \in D, \mathbf{y} \in \Gamma, \\ u(\mathbf{x}, \mathbf{y}) &= 0 & \mathbf{x} \in \partial D, \mathbf{y} \in \Gamma, \end{aligned} \tag{M}$$

where $D \subset \mathbb{R}^d$ ($d = 2, 3$), $\Gamma := \prod_{m=1}^{\infty} [-1, 1]$ and

$$a(\mathbf{x}, \mathbf{y}) = a_0(\mathbf{x}) + \sum_{m=1}^{\infty} a_m(\mathbf{x}) y_m, \quad \mathbf{x} \in D, \mathbf{y} \in \Gamma.$$

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General framework: parametric operator equation

$$A(\mathbf{y}) u(\mathbf{y}) = f(\mathbf{y}), \quad \text{where} \quad A(\mathbf{y}) = A_0 + \sum_{m=1}^{\infty} y_m A_m \quad \forall \mathbf{y} \in \Gamma,$$

where $A : \Gamma \rightarrow \mathcal{L}(H, H')$, $f : \Gamma \rightarrow H'$, H is a **separable** Hilbert space (cf. [Schwab, Gittelson '11]).

Parametric operator equation: weak formulation

$$A(\mathbf{y}) u(\mathbf{y}) = f(\mathbf{y}), \quad \text{where} \quad A(\mathbf{y}) = A_0 + \sum_{m=1}^{\infty} y_m A_m \quad \forall \mathbf{y} \in \Gamma.$$

Weak formulation: find $u \in V := L^2_{\pi}(\Gamma; H)$ such that

$$B(u, v) = F(v) \quad \forall v \in V, \quad (\star)$$

where

$$\begin{aligned} B(u, v) &= \int_{\Gamma} \langle A(\mathbf{y})u, v \rangle d\pi(\mathbf{y}) = \int_{\Gamma} \langle A_0 u, v \rangle d\pi(\mathbf{y}) + \sum_{m=1}^{\infty} \int_{\Gamma} \langle A_m u, v \rangle y_m d\pi(\mathbf{y}) \\ &=: B_0(u, v) + \sum_{m=1}^{\infty} B_m(u, v) \quad \forall u, v \in V, \end{aligned}$$

$$F(v) = \int_{\Gamma} \langle f(\mathbf{y}), v(\mathbf{y}) \rangle d\pi(\mathbf{y}).$$

Parametric operator equation: well posedness

$$A(\mathbf{y}) u(\mathbf{y}) = f(\mathbf{y}), \quad \text{where} \quad A(\mathbf{y}) = A_0 + \sum_{m=1}^{\infty} y_m A_m \quad \forall \mathbf{y} \in \Gamma.$$

Weak formulation: find $u \in V := L^2_{\pi}(\Gamma; H)$ such that

$$B_0(u, v) + \sum_{m=1}^{\infty} B_m(u, v) = F(v) \quad \forall v \in V. \quad (\star)$$

Assumptions ensuring the well posedness of (\star) :

- $A_0 \in \mathcal{L}(H, H')$ is symmetric and positive definite;
- $A_m \in \mathcal{L}(H, H')$ for $(m \in \mathbb{N})$ are symmetric and $\exists \tau \in [0, 1)$ s.t. $\forall \mathbf{y} \in \Gamma$

$$\left| \left\langle \sum_{m=1}^{\infty} y_m A_m v, v \right\rangle \right| \leq \tau \langle A_0 v, v \rangle \quad \forall v \in H$$

(cf. [Schwab, Gittelsohn '11]).

Parametric operator equation: norm equivalence

$$A(\mathbf{y}) u(\mathbf{y}) = f(\mathbf{y}), \quad \text{where} \quad A(\mathbf{y}) = A_0 + \sum_{m=1}^{\infty} y_m A_m \quad \forall \mathbf{y} \in \Gamma.$$

Weak formulation: find $u \in V := L^2_{\pi}(\Gamma; H)$ such that

$$\underbrace{B_0(u, v) + \sum_{m=1}^{\infty} B_m(u, v)}_{B(u, v)} = F(v) \quad \forall v \in V. \quad (\star)$$

Norm equivalence

Note that $(B(u, u))^{1/2} = \|u\|_B$ is **equivalent** to the B_0 -norm: there exist positive constants $\lambda < 1 < \Lambda$ such that

$$\lambda B(v, v) \leq B_0(v, v) \leq \Lambda B(v, v) \quad \forall v \in V. \quad (B_0)$$

Discrete formulation

Solution space: $V := L^2_\pi(\Gamma; H) \sim H \otimes L^2_\pi(\Gamma)$ (isometric isomorphism).

Finite-dimensional subspace: $X \otimes \mathcal{P}_\mathfrak{P} =: V_{X\mathfrak{P}} \subset V$

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Finite-dimensional subspace: $X \otimes \mathcal{P}_\mathfrak{P} =: V_{X\mathfrak{P}} \subset V$, where

- X is a finite-dimensional subspace of H
(corresponding, e.g., to spatial finite element discretization on D);
- $\mathcal{P}_\mathfrak{P} \subset L^2_\pi(\Gamma)$ is a polynomial space on Γ , more precisely, the set of tensor product polynomials associated with a **finite subset** \mathfrak{P} of the index set \mathfrak{I} of finitely supported sequences

$$\mathfrak{I} := \{ \nu = (\nu_1, \nu_2, \dots) \in \mathbb{N}_0^{\mathbb{N}}; \# \text{supp } \nu < \infty \},$$

where $\text{supp } \nu := \{m \in \mathbb{N}; \nu_m \neq 0\}$ for any $\nu \in \mathbb{N}_0^{\mathbb{N}}$.

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Galerkin projection: find $u_{X\mathfrak{P}} \in V_{X\mathfrak{P}} = X \otimes \mathcal{P}_\mathfrak{P}$ such that

$$B(u_{X\mathfrak{P}}, v) = F(v) \quad \forall v \in V_{X\mathfrak{P}}.$$

Linear algebra

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Discrete system: find $\mathbf{x} \in \mathbb{R}^{n_x \cdot n_\xi}$ such that $\mathcal{A}\mathbf{x} = \mathbf{b}$ with

$$\mathcal{A} = G_0 \otimes K_0 + \sum_{m=1}^M G_m \otimes K_m, \quad \mathbf{b} = \mathbf{g}_0 \otimes \mathbf{f}_0$$

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Latest reference:

Catherine Powell, David Silvester and Valeria Simoncini

An efficient reduced basis solver for stochastic Galerkin matrix equations.

MIMS Eprint 2015.64

<http://eprints.ma.man.ac.uk/2350/>

Software: Stochastic IFISS (S-IFISS) version 1.02, July 2015

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Error estimation I

$$u \in V : \quad B(u, v) = F(v) \quad \forall v \in V. \quad (\star)$$

$$u_{X\mathfrak{P}} \in V_{X\mathfrak{P}} = X \otimes \mathcal{P}_{\mathfrak{P}} : \quad B(u_{X\mathfrak{P}}, v) = F(v) \quad \forall v \in V_{X\mathfrak{P}}.$$

Thus, the error $e := u - u_{X\mathfrak{P}} \in V$ satisfies

$$B(e, v) = F(v) - B(u_{X\mathfrak{P}}, v) \quad \forall v \in V.$$

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Thus, the error $e := u - u_{X\mathfrak{P}} \in V$ satisfies

$$\begin{aligned} B(e, v) &= F(v) - \underbrace{B(u_{X\mathfrak{P}}, v)}_{= 0 \quad \forall v \in V_{X\mathfrak{P}}} \quad \forall v \in V. \end{aligned}$$

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This motivates the estimator $e^* \in V_{X\mathfrak{P}}^*$ satisfying

$$B(e^*, v) = F(v) - B(u_{X\mathfrak{P}}, v) \quad \forall v \in V_{X\mathfrak{P}}^*,$$

where $V_{X\mathfrak{P}}^* \supset V_{X\mathfrak{P}}$ is an **enriched** finite-dimensional subspace of V .

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Is the estimator e^* practical? **No.**

Error estimation II

The estimator $e^* \in V_{X\mathfrak{P}}^*$ satisfies: $B(e^*, v) = F(v) - B(u_{X\mathfrak{P}}, v) \quad \forall v \in V_{X\mathfrak{P}}^*$.

Recall that $B(u, v) = B_0(u, v) + \sum_{m=1}^{\infty} B_m(u, v)$.

Error estimation II

The estimator $e^* \in V_{X\mathfrak{T}}^*$ satisfies: $B(e^*, v) = F(v) - B(u_{X\mathfrak{T}}, v) \quad \forall v \in V_{X\mathfrak{T}}^*$.

Recall that $B(u, v) = B_0(u, v) + \sum_{m=1}^{\infty} B_m(u, v)$.

Exploit the structure of the **enriched** finite-dimensional space $V_{X\mathfrak{T}}^* \supset V_{X\mathfrak{T}}$:

$$X^* = X \oplus Y, \quad X \cap Y = \{0\}; \quad \mathcal{P}_{\mathfrak{T}^*} = \mathcal{P}_{\mathfrak{T}} \oplus \mathcal{P}_{\Omega}, \quad \mathcal{P}_{\mathfrak{T}} \cap \mathcal{P}_{\Omega} = \{0\};$$

$$V_{X\mathfrak{T}}^* := \underbrace{V_{X^*\mathfrak{T}}}_{X^* \otimes \mathcal{P}_{\mathfrak{T}}} \oplus \underbrace{V_{X\Omega}}_{X \otimes \mathcal{P}_{\Omega}} = \underbrace{V_{X\mathfrak{T}}}_{X \otimes \mathcal{P}_{\mathfrak{T}}} \oplus \underbrace{V_{Y\mathfrak{T}}}_{Y \otimes \mathcal{P}_{\mathfrak{T}}} \oplus \underbrace{V_{X\Omega}}_{X \otimes \mathcal{P}_{\Omega}}.$$

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This leads to the new error estimate

$$\eta := \left(\|e_{Y\mathfrak{T}}\|_{B_0}^2 + \|e_{X\Omega}\|_{B_0}^2 \right)^{1/2},$$

where two contributing estimators are defined by

$$e_{Y\mathfrak{T}} \in V_{Y\mathfrak{T}} : \quad B_0(e_{Y\mathfrak{T}}, v) = F(v) - B(u_{X\mathfrak{T}}, v) \quad \forall v \in V_{Y\mathfrak{T}},$$

$$e_{X\Omega} \in V_{X\Omega} : \quad B_0(e_{X\Omega}, v) = F(v) - B(u_{X\mathfrak{T}}, v) \quad \forall v \in V_{X\Omega}.$$

Error estimation III

The estimator $e^* \in V_{X\mathfrak{P}}^*$ satisfies: $B(e^*, v) = F(v) - B(u_{X\mathfrak{P}}, v) \quad \forall v \in V_{X\mathfrak{P}}^*$.

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$$e_{X\Omega} \in V_{X\Omega} : \quad B_0(e_{X\Omega}, v) = F(v) - B(u_{X\mathfrak{P}}, v) \quad \forall v \in V_{X\Omega}.$$

The relation between the estimator e^* and the componentwise estimate η :

$$\frac{\sqrt{\lambda}}{\sqrt{2}} \eta \leq \|e^*\|_B \leq \frac{\sqrt{\Lambda}}{\sqrt{1-\gamma^2}} \eta,$$

where λ, Λ are the norm-equivalence constants in (B_0) , $\gamma \in [0, 1)$ is the constant in the strengthened Cauchy–Schwarz inequality for the subspaces $X, Y \subset H$ w.r.t. inner product $\langle A_0 \cdot, \cdot \rangle$.

Error reduction indicators

Recall

$$\eta := \left(\|e_{Y\mathfrak{P}}\|_{B_0}^2 + \|e_{X\Omega}\|_{B_0}^2 \right)^{1/2},$$

with $e_{Y\mathfrak{P}} \in V_{Y\mathfrak{P}}$ and $e_{X\Omega} \in V_{X\Omega}$.

Enhanced Galerkin approximations: $u_{X^*\mathfrak{P}} \in X^* \otimes \mathcal{P}_{\mathfrak{P}}$ and $u_{X\mathfrak{P}^*} \in X \otimes \mathcal{P}_{\mathfrak{P}^*}$.

Error reduction indicators

Recall

$$\eta := \left(\|e_{Y\mathfrak{P}}\|_{B_0}^2 + \|e_{X\Omega}\|_{B_0}^2 \right)^{1/2},$$

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Enhanced Galerkin approximations: $u_{X^*\mathfrak{P}} \in X^* \otimes \mathcal{P}_{\mathfrak{P}}$ and $u_{X\mathfrak{P}^*} \in X \otimes \mathcal{P}_{\mathfrak{P}^*}$.

Estimates of the error reduction [Bespalov, Powell, Silvester '14]:

$$\sqrt{\lambda} \|e_{Y\mathfrak{P}}\|_{B_0} \leq \|u_{X^*\mathfrak{P}} - u_{X\mathfrak{P}}\|_B \leq \frac{\sqrt{\Lambda}}{\sqrt{1-\gamma^2}} \|e_{Y\mathfrak{P}}\|_{B_0},$$

$$\sqrt{\lambda} \|e_{X\Omega}\|_{B_0} \leq \|u_{X\mathfrak{P}^*} - u_{X\mathfrak{P}}\|_B \leq \sqrt{\Lambda} \|e_{X\Omega}\|_{B_0}.$$

Here, λ , Λ are the constants in (B_0) and $\gamma \in [0, 1)$ is the constant in the (deterministic) strengthened Cauchy–Schwarz inequality.

The error estimator $e_{X\Omega}$: decomposition

Lemma 1. For any *finite* detail set $\Omega = \{\mu \in \mathcal{I}; \mu \notin \mathfrak{P}\}$ one has

$$e_{X\Omega} = \sum_{\mu \in \Omega} e_{X\Omega}^{(\mu)} \quad \text{with} \quad \|e_{X\Omega}\|_{B_0}^2 = \sum_{\mu \in \Omega} \|e_{X\Omega}^{(\mu)}\|_{B_0}^2,$$

where $e_{X\Omega}^{(\mu)} \in X \otimes \mathcal{P}_\mu$ satisfies

$$B_0(e_{X\Omega}^{(\mu)}, v) = F(v) - B(u_{X\mathfrak{P}}, v) \quad \forall v \in X \otimes \mathcal{P}_\mu.$$

Note:

- each $e_{X\Omega}^{(\mu)}$ can be independently and cheaply computed (the same coefficient matrix for all $\mu \in \Omega$);
- $\|e_{X\Omega}^{(\mu)}\|_{B_0}$ provides the error reduction estimate for the corresponding enhanced approximation $u_{X\mathfrak{P}^*}$ with $\mathfrak{P}^* = \mathfrak{P} \cup \mu$.

The error estimator $e_{X\Omega}$: choosing Ω

Let $\mathcal{P}_{M,p}$ be the space of polynomials of total degree $\leq p$ in M parameters.

Dimension of $\mathcal{P}_{M,p}$: $\dim(\mathcal{P}_{M,p}) = \frac{(p+M)!}{p! M!}$.

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Example.

Fix the polynomial space on Γ to be $\mathcal{P}_{3,2}$, and consider an enriched polynomial space: $\mathcal{P}_{10,5} = \mathcal{P}_{3,2} \oplus \mathcal{P}_{\Omega}$.

Then $\dim(\mathcal{P}_{\Omega}) = \dim(\mathcal{P}_{10,5}) - \dim(\mathcal{P}_{3,2}) = 2993$.

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Then $\dim(\mathcal{P}_{\Omega}) = \dim(\mathcal{P}_{10,5}) - \dim(\mathcal{P}_{3,2}) = 2993$.

However, the number of indices $\mu \in \Omega$ for which the estimators $e_{X\Omega}^{(\mu)}$ are not zero in exact arithmetic is only 80.

The error estimator $e_{X\Omega}$: choosing Ω

Lemma 2. Assume that $f(\mathbf{y})$ has the decomposition

$$f(\mathbf{y}) = f_0 + \sum_{m=1}^{\infty} y_m f_m, \quad \forall \mathbf{y} \in \Gamma.$$

For any finite detail index set $\Omega \subset \mathcal{I}$ the error estimator $e_{X\Omega}^{(\mu)} \neq 0$ if

$$\mu \in \Omega_{\infty} := \left\{ \mu \in \mathcal{I} \setminus \mathfrak{P}; \mu = \nu \pm \varepsilon^{(m)}, \forall \nu \in \mathfrak{P}, \forall m = 1, 2, \dots \right\}.$$

Here, $\varepsilon^{(m)} = (\varepsilon_1^{(m)}, \varepsilon_2^{(m)}, \dots) \in \mathcal{I}$ represents the Kronecker delta sequence for the coordinate m , i.e., $\varepsilon_j^{(m)} = \delta_{jm}$ for any $j \in \mathbb{N}$.

Proof. Apply the three-term recurrence formula for orthogonal polynomials.

Numerical experiments: model problem

[Eigel, Gittelsohn, Schwab, Zander '14]

We solve the steady-state diffusion problem

$$\begin{aligned} -\nabla \cdot (a(\mathbf{x}, \mathbf{y}) \nabla u(\mathbf{x}, \mathbf{y})) &= f(\mathbf{x}) & \mathbf{x} \in D, \mathbf{y} \in \Gamma, \\ u(\mathbf{x}, \mathbf{y}) &= 0 & \mathbf{x} \in \partial D, \mathbf{y} \in \Gamma \end{aligned} \quad (M)$$

with $\mathbf{x} = (x_1, x_2) \in D := [0, 1]^2$, $\mathbf{y} \in \Gamma = \prod_{m=1}^{\infty} [-1, 1]$, $f(\mathbf{x}) = 1$ and

$$a(\mathbf{x}, \mathbf{y}) = \underbrace{1}_{a_0 = \mathbb{E}[a]} + \sum_{m=1}^{\infty} \bar{\alpha} m^{-\tilde{\sigma}} \cos(2\pi\beta_1(m)x_1) \cos(2\pi\beta_2(m)x_2) y_m.$$

Here $\tilde{\sigma} > 1$ and $0 < \bar{\alpha} < 1/\zeta(\tilde{\sigma})$ (here, ζ denotes the Riemann zeta function);

y_m are the images of uniformly distributed independent mean-zero r.v.

The weak formulation (\star) of this problem is well posed because $\tau = \bar{\alpha}\zeta(\tilde{\sigma}) < 1$.

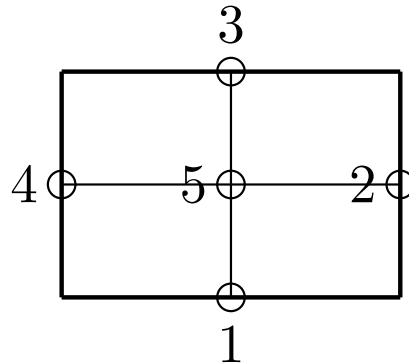
Two cases are considered: $\tilde{\sigma} = 4$ (fast decay) and $\tilde{\sigma} = 2$ (slow decay).

model problem | spatial detail space

The detail spaces $Y(h_k)$ for bilinear (Q_1) spatial approximation satisfy

$$X_{h/2} = X_h \oplus Y_h.$$

Thus, Y_h spans the set of bilinear bubble functions corresponding to the edge midpoints and element centroids of the original mesh \square_h (defined locally)



This gives a decomposition of the error estimate η with

$$V_{X\mathfrak{P}}^* := V_{X\mathfrak{P}} \oplus (Y_h \otimes \mathcal{P}_{\mathfrak{P}}) \oplus (X_h \otimes \mathcal{P}_{\Omega}).$$

The smart choice of the detail space Y_h gives rise to **local problems** defined on **all elements** $K \in \square_h$.

model problem | spatial detail space

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$$V_{X\mathfrak{P}}^* := V_{X\mathfrak{P}} \oplus (Y_h \otimes \mathcal{P}_{\mathfrak{P}}) \oplus (X_h \otimes \mathcal{P}_{\Omega}).$$

For example, we compute $\bar{e}_{Y\mathfrak{P}}|_K \in Y_h|_K \otimes \mathcal{P}_{\mathfrak{P}}$ satisfying

$$\begin{aligned} B_{0,K}(\bar{e}_{Y\mathfrak{P}}|_K, v) &= F_K(v) \\ &+ \int_{\Gamma} \int_K \nabla \cdot (a(\mathbf{x}, \mathbf{y}) \nabla u_{X\mathfrak{P}}(\mathbf{x}, \mathbf{y})) v(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\pi(\mathbf{y}) \\ &- \frac{1}{2} \int_{\Gamma} \int_{\partial K \setminus \partial D} a(s, \mathbf{y}) \left[\frac{\partial u_{X\mathfrak{P}}}{\partial n} \right] v(s, \mathbf{y}) \, ds \, d\pi(\mathbf{y}), \end{aligned}$$

for all $v \in Y_h|_K \otimes \mathcal{P}_{\mathfrak{P}}$.

The coefficient matrix associated with this local problem is the Kronecker product of a 5×5 (stiffness) matrix and an identity matrix of dimension $|P| = \dim(\mathcal{P}_{\mathfrak{P}})$.

Adaptive algorithm

Adaptive_sGFEM [tol, A, f] $\rightarrow u_n$

input h_0, \mathfrak{P}_0

for $k = 0, 1, 2, \dots$ **do**

$u_k \leftarrow$ Solve [$A, f, X(h_k), \mathfrak{P}_k$]

$\delta_X \leftarrow$ Error_Estimate_1 [$A, f, u_k, Y(h_k)$]

$\Omega_k \leftarrow$ Detail_Index_Set [\mathfrak{P}_k]

for $i = 1, 2, \dots, \#(\Omega_k)$ **do**

$\delta_{\mathfrak{P},i} \leftarrow$ Error_Estimate_2 [A, f, u_k, μ_i]

end

$\eta_k := \left(\delta_X^2 + \sum_{i=1}^{\#(\Omega_k)} \delta_{\mathfrak{P},i}^2 \right)^{1/2}$

if $\eta_k < tol$ **then** $n := k$, **break**

if $\delta_X \geq \max \{ \delta_{\mathfrak{P},i}; i = 1, 2, \dots, \#(\Omega_k) \}$ **then**

$X(h_{k+1}) := X(h_k) \oplus Z(h_{k+1}), \mathfrak{P}_{k+1} := \mathfrak{P}_k$

else $X(h_{k+1}) := X(h_k), \mathfrak{P}_{k+1} := \mathfrak{P}_k \cup \{ \mu_i \in \Omega_k; \delta_{\mathfrak{P},i} \geq \delta_X \}$

end

Adaptive algorithm

Adaptive_sGFEM [tol, A, f] $\rightarrow u_n$

input h_0, \mathfrak{P}_0

for $k = 0, 1, 2, \dots$ **do**

$u_k \leftarrow$ Solve [$A, f, X(h_k), \mathfrak{P}_k$]

$\delta_X \leftarrow$ Error_Estimate_1 [$A, f, u_k, Y(h_k)$]

$\Omega_k \leftarrow$ Detail_Index_Set [\mathfrak{P}_k]

% Lemma 2

for $i = 1, 2, \dots, \#(\Omega_k)$ **do**

$\delta_{\mathfrak{P},i} \leftarrow$ Error_Estimate_2 [A, f, u_k, μ_i]

end

$\eta_k := \left(\delta_X^2 + \sum_{i=1}^{\#(\Omega_k)} \delta_{\mathfrak{P},i}^2 \right)^{1/2}$

if $\eta_k < tol$ **then** $n := k$, **break**

if $\delta_X \geq \max \{ \delta_{\mathfrak{P},i}; i = 1, 2, \dots, \#(\Omega_k) \}$ **then**

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end

Adaptive algorithm

Adaptive_sGFEM [tol, A, f] $\rightarrow u_n$

input h_0, \mathfrak{P}_0

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$\Omega_k \leftarrow$ Detail_Index_Set [\mathfrak{P}_k]

% Lemma 2

for $i = 1, 2, \dots, \#(\Omega_k)$ **do**

$\delta_{\mathfrak{P},i} \leftarrow$ Error_Estimate_2 [A, f, u_k, μ_i]

end

$\eta_k := \left(\delta_X^2 + \sum_{i=1}^{\#(\Omega_k)} \delta_{\mathfrak{P},i}^2 \right)^{1/2}$

% Lemma 1

if $\eta_k < tol$ **then** $n := k$, **break**

if $\delta_X \geq \max \{ \delta_{\mathfrak{P},i}; i = 1, 2, \dots, \#(\Omega_k) \}$ **then**

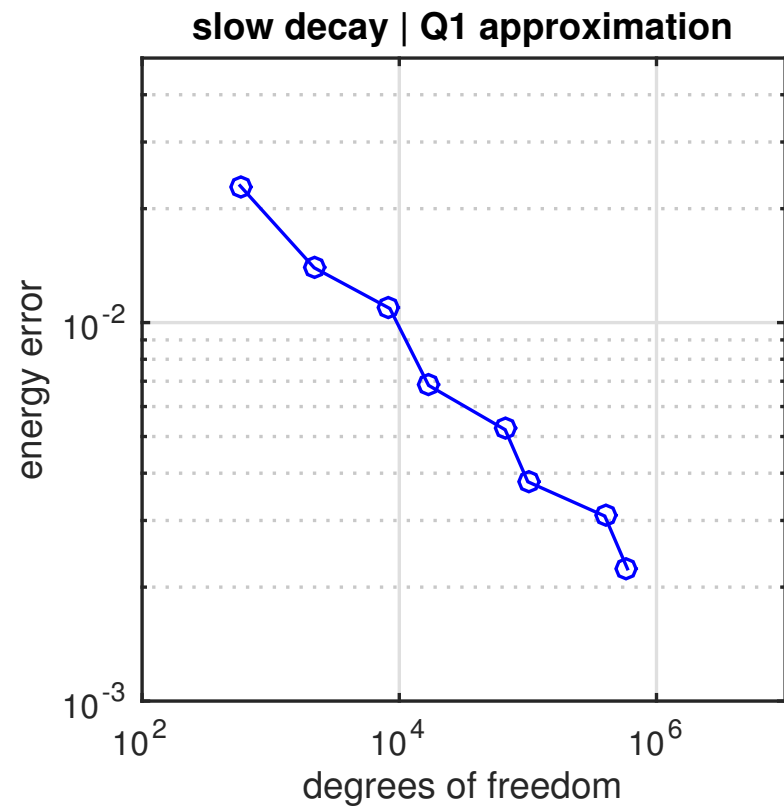
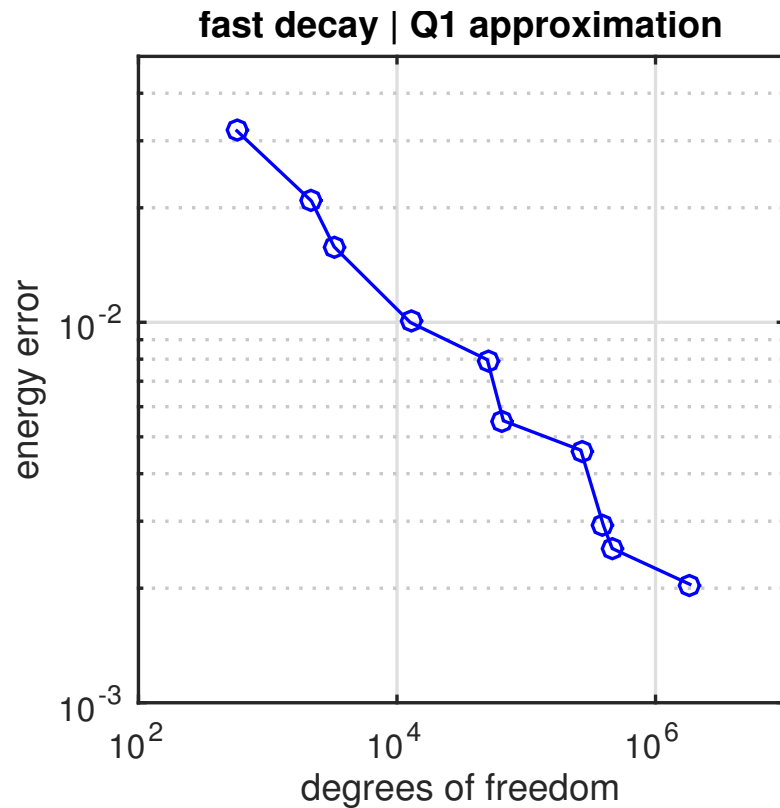
$X(h_{k+1}) := X(h_k) \oplus Z(h_{k+1}), \mathfrak{P}_{k+1} := \mathfrak{P}_k$

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end

Numerical experiments I: performance of the adaptive algorithm

Bilinear (Q_1) spatial approximation on uniform grids \square_h



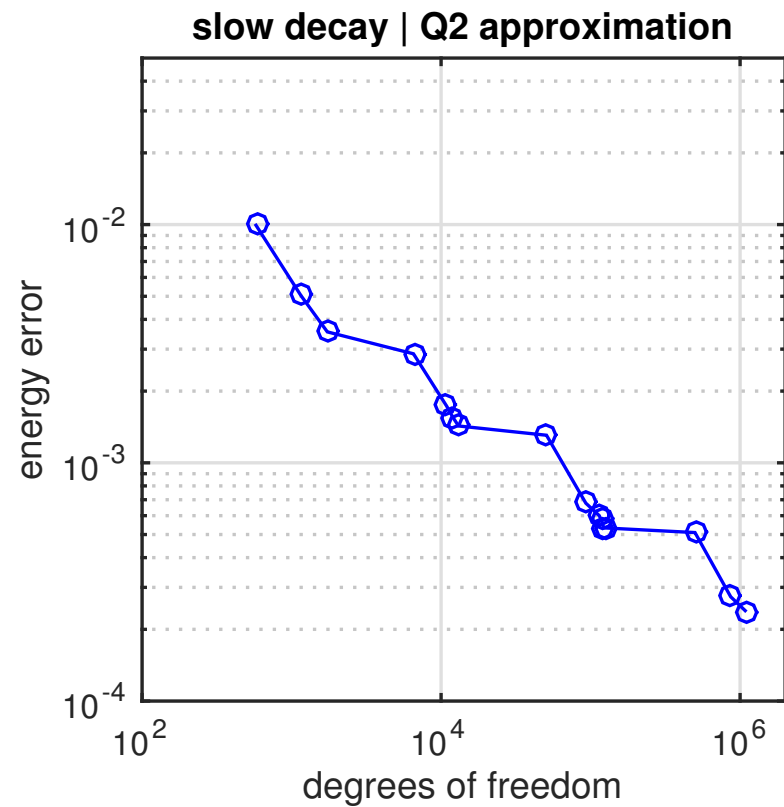
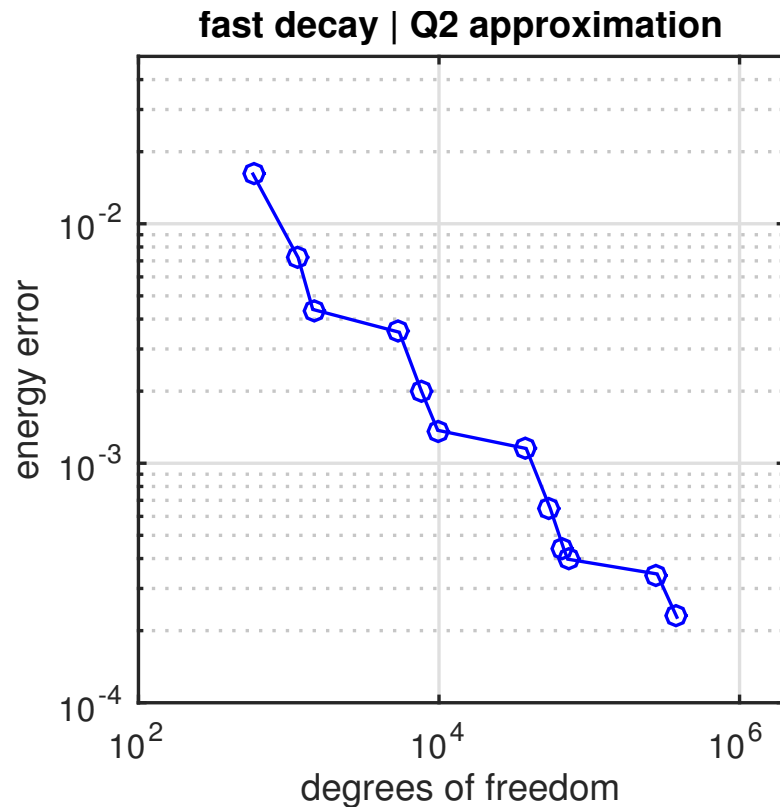
Numerical experiments II: evolution of the index set

Bilinear (Q_1) spatial approximation on uniform grids \square_h

k	fast decay	slow decay
0	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
1	—	—
2	$\begin{pmatrix} 2 & 0 \end{pmatrix}$	—
3	—	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$
4	—	—
5	$\begin{pmatrix} 3 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$
6	—	—
7	$\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$
8	$\begin{pmatrix} 1 & 1 \end{pmatrix}$	
9	—	

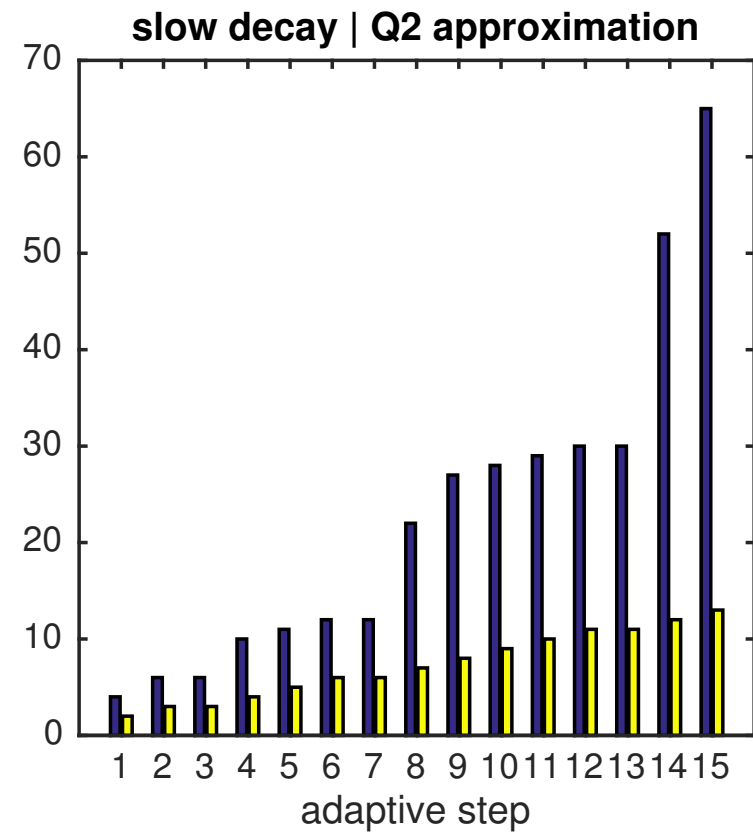
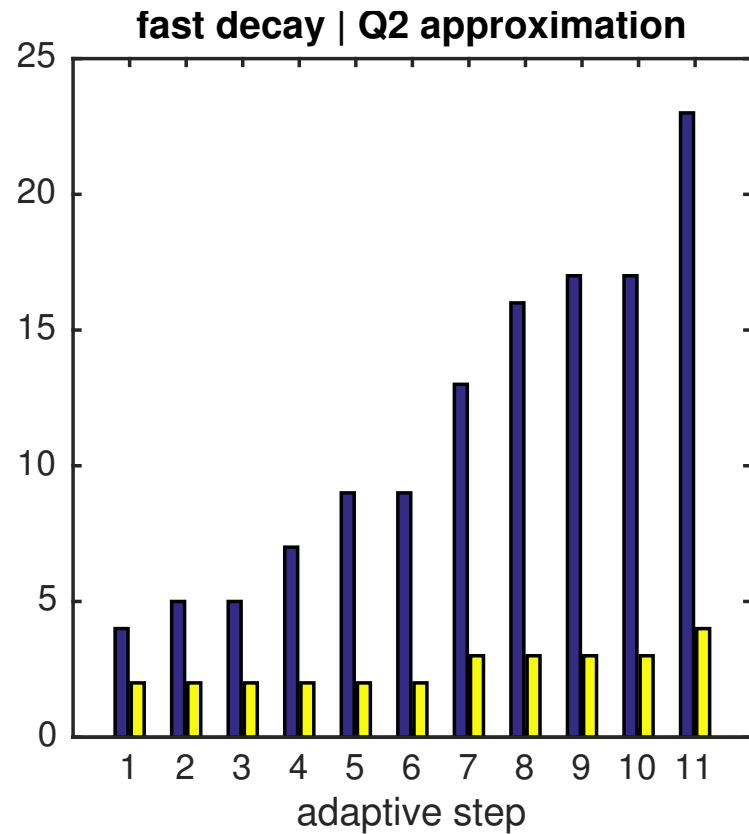
Numerical experiments III: performance of the adaptive algorithm

Biquadratic (Q_2) spatial approximation on uniform grids \square_h



Numerical experiments IV: evolution of the index set

Biquadratic (Q_2) spatial approximation on uniform grids \square_h



Numerical experiments V: effectivity

$u \in L^2_\pi(\Gamma, H_0^1(D))$ – the (unknown) exact solution;

$u_k \in X_k \otimes \mathcal{P}_{\mathfrak{P}_k}$ – the Galerkin solution at the step $k = 0, 1, 2, \dots$;

$e_k := u - u_k$ – the true error ($\|e_k\|_B \approx \eta_k$).

The error representation:

$$\begin{aligned}\|e_k\|_B &= \|u - u_k\|_B = \left(\|u\|_B^2 - \|u_k\|_B^2 \right)^{1/2} \\ &\approx \left(\|u_{\text{ref}}\|_B^2 - \|u_k\|_B^2 \right)^{1/2} := \|e_k^{\text{ref}}\|_B,\end{aligned}$$

where $u_{\text{ref}} \in X_{\text{ref}} \otimes \mathcal{P}_{\mathfrak{P}_{\text{ref}}}$ is an accurate (reference) solution.

The effectivity index:

$$\theta_k = \frac{\eta_k}{\|e_k^{\text{ref}}\|_B}.$$

In the case of Q_1 approximations:

u_{ref} is generated by running the adaptive algorithm with Q_2 approximations on a fine spatial grid until a small tolerance ($\text{tol} = 2.5\text{e-}4$) is reached.

Numerical experiments V: effectivity

Model problem with $\tilde{\sigma} = 2$ (slow decay).

The Galerkin solution u_k using Q_1 approximation ($k = 0, 1, 2, \dots$).

The reference solution u_{ref} using Q_2 approximation

$$\|u_{\text{ref}}\|_B = 1.90117\text{e-}01, N_{\text{ref}} = 1'081'665.$$

$$\text{effectivity indices: } \theta_k = \eta_k / \|e_k^{\text{ref}}\|_B.$$

k	N_k	$\ u_k\ _B$	η_k	$\ e_k^{\text{ref}}\ _B$	θ_k
0	578	1.89179e-01	2.29925e-02	1.88606e-02	1.22
1	2178	1.89633e-01	1.39596e-02	1.35540e-02	1.03
2	8450	1.89746e-01	1.08927e-02	1.18594e-02	0.92
3	16900	1.89996e-01	6.82414e-03	6.76065e-03	1.01
4	66564	1.90025e-01	5.21155e-03	5.89584e-03	0.88
5	99846	1.90074e-01	3.79683e-03	4.03873e-03	0.94
6	396294	1.90081e-01	3.08163e-03	3.68263e-03	0.84
7	594441	1.90100e-01	2.23643e-03	2.55596e-03	0.87

Conclusions

What have we achieved?

- ★ **Novel error estimation strategy:** this exploits the tensor product structure of the approximation spaces.
- ★ **Effective adaptive algorithm:** adaptive refinement is driven by dominant error reduction estimates.

Conclusions

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- ★ **Novel error estimation strategy:** this exploits the tensor product structure of the approximation spaces.
- ★ **Effective adaptive algorithm:** adaptive refinement is driven by dominant error reduction estimates.

Further reading:

Alex Bespalov & David Silvester, [Efficient adaptive stochastic Galerkin methods for parametric operator equations](http://eprints.ma.man.ac.uk/2313/), MIMS Eprint 2015.42, University of Manchester, 2015. <http://eprints.ma.man.ac.uk/2313/>