Adaptive algorithms for computational PDEs

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Axioms of Adaptivity

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joint work with

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Introduction

Adaptive Algorithm

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \leq 1$

For all $\ell = 0, 1, 2, 3, \dots$ iterate

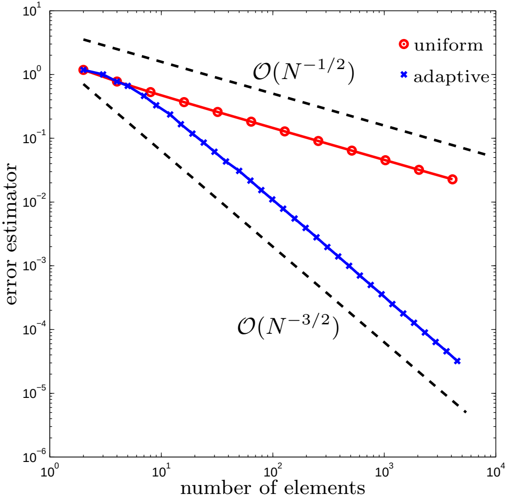
1. SOLVE: compute discrete solution U_ℓ for mesh $\mathcal{T}_\ell$
2. ESTIMATE: compute indicators $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$
3. MARK: find (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

$$\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2$$

4. REFINE: refine (at least) all $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$

Dörfler: SINUM 33 (1996)

What is all about?

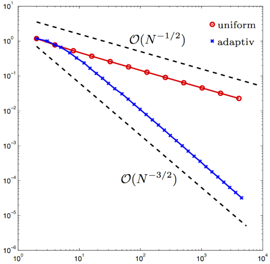


Feischl, Karkulik, Melenk, Praetorius: SINUM 51 (2013)

Gantumur: Numer. Math. 123 (2013)

Mathematical Questions

- can we prove convergence of algorithm?
- can we guarantee optimal convergence rates?
  - at least asymptotically
- what problem class can be covered?
  - AFEM for 2nd order elliptic PDEs?
  - ABEM for 2nd order elliptic PDEs?
  - linear/nonlinear problems?
  - goal-oriented adaptivity?



Dörfler: SINUM 33 (1996) 324 citations

Morin, Nochetto, Siebert: SINUM 38 (2000) 184 citations

Binev, Dahmen, DeVore: Numer. Math. 97 (2004) 176 citations

Stevenson: Found. Comput. Math. 7 (2007) 134 citations

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008) 140 citations

Axioms of Adaptivity?

Carstensen, Feischl, Page, P. '14

1. reproduces all results on rate optimality of adaptive algorithms
   - independent of linear or nonlinear problem
   - independent of discretization (e.g., FEM, BEM, FVM, coupled)
   - equivalent estimators (not only residual estimators)
   - inexact solvers
2. four properties $(=$ axioms$)$ of error estimator are sufficient
   - two axioms are even necessary
3. problem $+$ discretization enter only through proof of axioms

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Outline

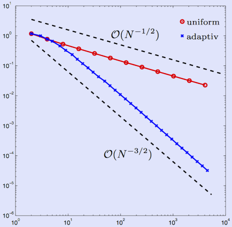
Axioms of Adaptivity

Main Theorem on Adaptive Algorithms

Theorem (Stevenson '07, ..., Carstensen, Feischl, Page, P. '14)

- validity of axioms (A1)–(A4)
- $0 < \theta \le 1$

$\Longrightarrow \quad \exists C > 0 \ \exists 0 < q < 1 \ \forall \ell, n \ge 0 \quad \eta_{\ell+n} \le C\, q^n\, \eta_\ell$

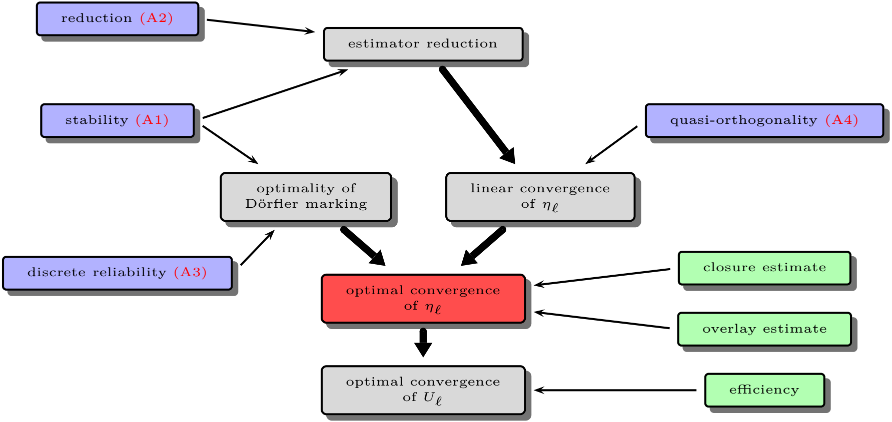


- $\mathbb{T}_N := \{\mathcal{T} \in \texttt{refine}(\mathcal{T}_0) \,:\, \#\mathcal{T} \le N\} \cup \{\mathcal{T}_0\}$
- $s > 0$ arbitrary
- $0 < \theta \ll 1$ sufficiently small
- $\mathcal{M}_\ell$ has (essentially) minimal cardinality

$$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell \simeq \sup_{N>0} \Big(N^s \min_{\mathcal{T}_{\rm opt} \in \mathbb{T}_N} \eta_{\rm opt}\Big) =: \|\eta\|_{\mathbb{A}_s}$$

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Axioms of Adaptivity



Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

The Axioms

$\forall \mathcal{T}_+ \quad \forall \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_+)$

(A1) $$\Big| \Big(\sum_{T \in \mathcal{T}_+ \cap \mathcal{T}_\star} \eta_\star(T)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T}_+ \cap \mathcal{T}_\star} \eta_+(T)^2 \Big)^{1/2} \Big| \le C_{\rm stab} \, |\!|\!| U_\star - U_+ |\!|\!|$$

(A2) $$\sum_{T \in \mathcal{T}_\star \setminus \mathcal{T}_+} \eta_\star(T)^2 \le q_{\rm red} \sum_{T \in \mathcal{T}_+ \setminus \mathcal{T}_\star} \eta_+(T)^2 + C_{\rm red} \, |\!|\!| U_\star - U_+ |\!|\!|^2$$

(A3) $$|\!|\!| U_\star - U_+ |\!|\!|^2 \le C_{\rm rel}^2 \sum_{T \in \mathcal{R}_+} \eta_+(T)^2$$

where $\mathcal{T}_+ \setminus \mathcal{T}_\star \subseteq \mathcal{R}_+ \subseteq \mathcal{T}_+, \quad \#\mathcal{R}_+ \le C_{\rm rel} \, \#(\mathcal{T}_+ \setminus \mathcal{T}_\star)$

$\forall \ell, N \ge 0 \quad \forall \varepsilon > 0$

(A4) $$\sum_{k=\ell}^{N} \big(|\!|\!| U_{k+1} - U_k |\!|\!|^2 - \varepsilon \eta_k^2 \big) \le C_{\rm orth}(\varepsilon) \, \eta_\ell^2$$

Poisson Model Problem

Strong formulation

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = 0 \quad \text{on } \Gamma = \partial\Omega$$

Weak formulation

- find $u \in H^1_0(\Omega)$ s.t.

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \text{for all } v \in H^1_0(\Omega)$$

Residual Error Estimator for Poisson Model Problem

Reliability and efficiency

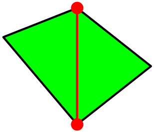
- $|||u - U_\star||| \lesssim \eta_\star \lesssim |||u - U_\star||| + \mathrm{osc}_\star$

- $|||\cdot||| = \|\nabla(\cdot)\|_{L^2(\Omega)}$

- $\eta_\star = \Big(\sum_{T\in\mathcal{T}_\star} \eta_\star(T)^2\Big)^{1/2}$

- $\eta_\star(T)^2 = h_T^2\,\|f\|_{L^2(T)}^2 + h_T\,\|[\partial_n U_\star]\|_{L^2(\partial T\cap\Omega)}^2$

- $\mathrm{osc}_\star := \Big(\sum_{T\in\mathcal{T}_\star} h_T^2\,\|f - f_T\|_{L^2(T)}^2\Big)^{1/2}$



Axiom (A1): Stability on Non-Refined Elements

(A1) Stability on non-refined elements, $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_+)$

$$\Big| \Big(\sum_{T \in \mathcal{T}_+ \cap \mathcal{T}_\star} \eta_\star(T)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T}_+ \cap \mathcal{T}_\star} \eta_+(T)^2 \Big)^{1/2} \Big| \le C_{\rm stab} \, |||U_\star - U_+|||$$

- verification for Poisson model problem:
- $\eta_\star(T)^2 = h_T^2 \, \|f\|^2_{L^2(T)} + h_T \, \|[\partial_n U_\star]\|^2_{L^2(\partial T \cap \Omega)}$
- inverse triangle inequality + scaling arguments

$$\text{LHS} \le \Big(\sum_{T \in \mathcal{T}_+ \cap \mathcal{T}_\star} h_T \, \|[\partial_n (U_\star - U_+)]\|^2_{L^2(\partial T \cap \Omega)} \Big)^{1/2} \lesssim \|\nabla (U_\star - U_+)\|_{L^2(\Omega)}$$

Cascón, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Axiom (A2): Reduction on Refined Elements

(A2) Reduction on refined elements, $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_+)$

$$\sum_{T\in\mathcal{T}_\star\backslash\mathcal{T}_+} \eta_\star(T)^2 \le q_{\rm red} \sum_{T\in\mathcal{T}_+\backslash\mathcal{T}_\star} \eta_+(T)^2 + C_{\rm red}\, |||U_\star - U_+|||^2$$

- verification for Poisson model problem:
- $\eta_\star(T)^2 = h_T^2\,\|f\|_{L^2(T)}^2 + h_T\,\|[\partial_n U_\star]\|_{L^2(\partial T\cap\Omega)}^2$
- $\bigcup(\mathcal{T}_\star\backslash\mathcal{T}_+) = \bigcup(\mathcal{T}_+\backslash\mathcal{T}_\star)$
- $h_{T'} \le \frac{1}{2}\,h_T$ for $\mathcal{T}_\star \ni T' \subsetneqq T \in \mathcal{T}_+$

- triangle inequality + Young inequality + scaling arguments
- $q_{\rm red} \approx \frac{1}{2}$

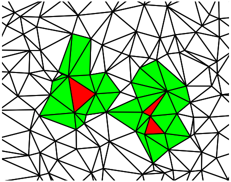
Cascón, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Axiom (A3): Discrete Reliability

(A3) Discrete reliability, $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_+)$

- exists $\mathcal{R}_+ \subseteq \mathcal{T}_+$ with
  - $\mathcal{T}_+ \backslash \mathcal{T}_\star \subseteq \mathcal{R}_+$
  - $\#\mathcal{R}_+ \leq C_{\rm rel} \#(\mathcal{T}_+ \backslash \mathcal{T}_\star)$
  - $\|U_\star - U_+\|^2 \leq C_{\rm rel}^2 \sum_{T \in \mathcal{R}_+} \eta_+(T)^2$

- discrete reliability $\Longrightarrow$ reliability
- $\mathcal{R}_+ = \mathcal{T}_+ \backslash \mathcal{T}_\star$ for FEM
- $\mathcal{R}_+ = \mathrm{patch}(\mathcal{T}_+ \backslash \mathcal{T}_\star)$ for BEM / FVM



Stevenson: Found. Comput. Math. 7 (2007)

Axiom (A4): Quasi-Orthogonality

(A4) Quasi-orthogonality, for all $\varepsilon > 0$ and ℓ, N

$$\sum_{k=\ell}^{N} \big(|||U_{k+1} - U_k|||^2 - \varepsilon \eta_k^2 \big) \le C_{\rm orth}(\varepsilon)\, \eta_\ell^2$$

- verification for Poisson model problem
- Galerkin orthogonality + symmetry $\implies$ Pythagoras theorem

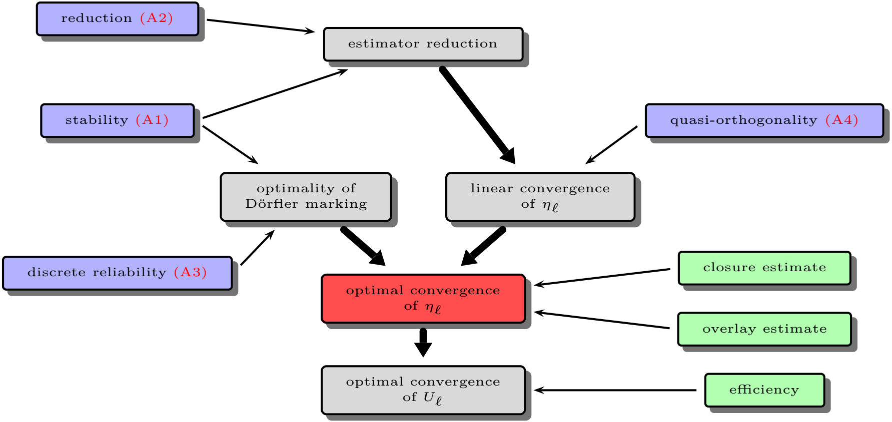
$$|||u - U_{k+1}|||^2 + |||U_{k+1} - U_k|||^2 = |||u - U_k|||^2$$

- telescoping series $\implies$ quasi-orth. with $C_{\rm orth}(\varepsilon) = C_{\rm rel}^2$, $\varepsilon = 0$

$$\sum_{k=\ell}^{N} |||U_{k+1} - U_k|||^2 = \sum_{k=\ell}^{N} \big(|||u - U_k|||^2 - |||u - U_{k+1}|||^2 \big) \le |||u - U_\ell|||^2$$

Feischl, Führer, Praetorius: SINUM 52 (2014)

Axioms of Adaptivity



Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Optimal Standard Adaptivity

Estimator Reduction

Stability (A1) + Reduction (A2) $\Longrightarrow$ Estimator Reduction

- $\forall 0 < \theta \le 1 \;\; \exists 0 < q_{\rm est} < 1 \;\; \exists C_{\rm est} > 0 \;\; \forall \ell \in \mathbb{N}_0$:

$$\eta_{\ell+1}^2 \le q_{\rm est}\, \eta_\ell^2 + C_{\rm est}\, \| U_{\ell+1} - U_\ell \|^2$$

- sketch: Young inequality + (A1) + (A2) + Dörfler marking
- $q_{\rm est} = (1+\delta) - \theta(1+\delta-q_{\rm red}) \approx 1 - \theta/2$
- $C_{\rm est} = C_{\rm stab}^2(1+\delta^{-1}) + C_{\rm red}$

Cascón, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Linear Convergence $\Longrightarrow$ Quasi-Orthogonality

Proposition (Carstensen, Feischl, Page, P. 14)

- reliability $|\!|\!| u - U_\ell |\!|\!| \lesssim \eta_\ell$
- linear convergence $\eta_{\ell+n} \le C_{\rm lin}\, q_{\rm lin}^n\, \eta_\ell$

$\Longrightarrow$ quasi-orthogonality (A4) with $\varepsilon = 0$, $C_{\rm orth}(\varepsilon) = C_{\rm orth}(0) > 0$

- sketch: triangle inequality + reliability + linear convergence

$$\Longrightarrow \quad \sum_{k=\ell}^{N} |\!|\!| U_{k+1} - U_k |\!|\!|^2 \lesssim \sum_{k=\ell}^{N+1} |\!|\!| u - U_k |\!|\!|^2 \lesssim \sum_{k=\ell}^{\infty} \eta_k^2 \lesssim \eta_\ell^2$$

Linear Convergence $\Longleftarrow$ Quasi-Orthogonality

Proposition (Carstensen, Feischl, Page, P. 14)

- estimator reduction for $0 < \theta \le 1$, e.g., stab. (A1) + red. (A2)
- quasi-orthogonality (A4)

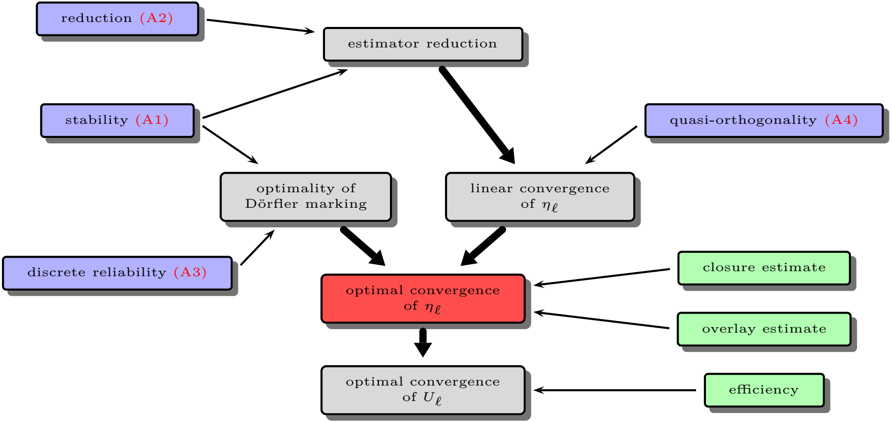
$\Longrightarrow$ linear convergence $\eta_{\ell+n} \le C_{\rm lin}\, q_{\rm lin}^n\, \eta_\ell$

- sketch: estimator reduction + quasi-orthogonality (A4)

$$\Longrightarrow \quad \sum_{k=\ell+1}^{\infty} \eta_k^2 \lesssim \eta_\ell^2$$

- basic calculus $\Longrightarrow$ equivalence to linear convergence

Road Map



Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Dörfler Marking $\Longrightarrow$ Discrete Reliability (A3)

- suppose Poisson model problem
- $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_{\ell+1})$

$\Longrightarrow$ $|||U_\star - U_\ell|||^2 \le |||u - U_\ell|||^2$

- reliability & Dörfler marking & $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \backslash \mathcal{T}_\star$

$\Longrightarrow$ $|||u - U_\ell|||^2 \lesssim \eta_\ell^2 \le \theta^{-1} \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2 \le \theta^{-1} \sum_{T \in \mathcal{T}_\ell \backslash \mathcal{T}_\star} \eta_\ell(T)^2$

$\Longrightarrow$ discrete reliability (A3)

$$|||U_\star - U_\ell|||^2 \lesssim \sum_{T \in \mathcal{T}_\ell \backslash \mathcal{T}_\star} \eta_\ell(T)^2$$

Dörfler Marking $\Longleftarrow$ Convergence + Discrete Rel. (A3)

- Dörfler + (A1) + (A2) + (A4) $\Longrightarrow$ linear conv. $\eta_{\ell+n} \lesssim q_{\rm lin}^n \, \eta_\ell$

Stab. (A1) + Rel. (A3) $\Longrightarrow$ Optimality of Dörfler Marking

For $0 < \theta < \theta_{\rm opt} := (1 + C_{\rm stab}^2 C_{\rm rel}^2)^{-1} < 1$, exists $0 < q_{\rm opt} < 1$ s.t.

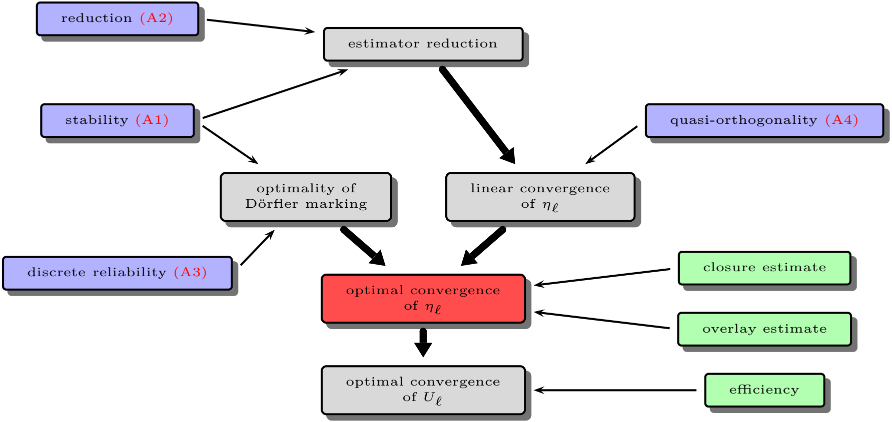
- for all $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ with $\eta_\star^2 \le q_{\rm opt}\, \eta_\ell^2$
- and $\mathcal{R}_\ell$ from discrete reliability (A3)

holds Dörfler marking $\theta \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 \le \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2$

- linear convergence

$\Longrightarrow$ Dörfler marking holds every fixed number n of steps

Road Map



Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Optimal Convergence Rates

Theorem (Stevenson '07, ..., Carstensen, Feischl, Page, P. '14)

- validity of axioms (A1)–(A4)
- "optimal mesh-refinement"
- $s > 0$ arbitrary
- $0 < \theta < \theta_{\rm opt} := (1 + C_{\rm stab}^2 C_{\rm rel}^2)^{-1}$
- $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ has (essentially) minimal cardinality

$$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell \simeq \sup_{N>0} \Big(N^s \min_{\mathcal{T}_{\rm opt} \in \mathbb{T}_N} \eta_{\rm opt} \Big) =: \|\eta\|_{\mathbb{A}_s}$$

- proof analyzes AFEM proof of Stevenson '07
  1. linear convergence $\eta_{\ell+n} \lesssim q_{\rm lin}^n \, \eta_\ell$ (sufficient and required)
  2. optimality of Dörfler marking, e.g., stab. (A1) and rel. (A3)
  3. $\rightsquigarrow$ closure estimate & overlay estimate
  4. efficiency is not needed

Optimal Computational Effort / Time

- suppose computational effort $E(\mathcal{T}) \simeq (\#\mathcal{T})^\gamma$ for some $\gamma \geq 1$

$\Longrightarrow$ adaptive mesh $\mathcal{T}_\ell$ requires cumulative effort $\sum_{k=0}^{\ell} E(\mathcal{T}_k)$

Corollary (Feischl '15)

- suppose assumptions of main theorem, e.g., $s > 0$ arbitrary
- $\widetilde{\mathcal{T}}_{\ell+1}$ sequence of meshes, obtained by successive refinement

$$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} \Big(\sum_{k=0}^{\ell} E(\mathcal{T}_k) \Big)^s \eta_\ell \lesssim \sup_{\ell \in \mathbb{N}_0} \big(E(\widetilde{\mathcal{T}}_\ell)^s \, \widetilde{\eta}_\ell \big)$$

- i.e., if rate s is possible for η w.r.t. computational effort / time

$\Longrightarrow$ adaptivity guarantees rate s even w.r.t. cumulative effort / time

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Feischl: PhD thesis, TU Wien (2015)

Optimal Goal-Oriented Adaptivity

Goal-Oriented Adaptivity 1/3

- $\mathcal{H}$ Hilbert space
- $a(\cdot,\cdot)$ continuous, elliptic, bilinear
- $f, q \in \mathcal{H}^*$ continuous, linear
- only interested in quantity of interest $q(u)$, where

$$a(u,v) = f(v) \quad \text{for all } v \in \mathcal{H}$$

- $U_\star \in \mathcal{X}_\star \le \mathcal{H}$ approximation of u

$\Longrightarrow$ naive error estimate $|q(u) - q(U_\star)| \lesssim |||u - U_\star||| \lesssim N^{-s}$

- standard adaptivity with $\|\eta\|_{\mathbb{A}_s} < \infty$

Goal-Oriented Adaptivity 2/3

- recall primal problem

$$a(u,v) = f(v) \quad \text{for all } v \in \mathcal{H}$$

- use Galerkin approximation

$$a(U_\star, V_\star) = f(V_\star) \quad \text{for all } V_\star \in \mathcal{X}_\star$$

- consider dual problem

$$a(v,z) = q(v) \quad \text{for all } v \in \mathcal{H}$$

- $Z_\star \in \mathcal{X}_\star$ approximation of z yields

$$q(u) - q(U_\star) = q(u - U_\star) = a(u - U_\star, z) = a(u - U_\star, z - Z_\star)$$

$\Longrightarrow$ improved estimate $|q(u) - q(U_\star)| \lesssim |||u - U_\star||| \, |||z - Z_\star|||$

- aim: possible $|q(u) - q(U_\ell)| = \mathcal{O}(N^{-(s+t)})$

Goal-Oriented Adaptivity 3/3

- suppose error estimators $\eta_{z,\star}$, $\eta_{u,\star}$ with (A1)–(A4)

$$\Longrightarrow \quad |q(u) - q(U_\star)| \lesssim |||u - U_\star||| \, |||z - Z_\star||| \lesssim \eta_{u,\star}\eta_{z,\star}$$

- aim: design optimal algorithm for estimator product
- had only been addressed for Poisson model problem

Mommer, Stevenson: SINUM 47 (2009)

Becker, Estecahandy, Trujillo: SINUM 49 (2011)

Goal-Oriented Adaptive Algorithm

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \leq 1$

For all $\ell = 0, 1, 2, 3, \ldots$ iterate

1. SOLVE: compute discrete solutions U_ℓ, Z_ℓ for mesh $\mathcal{T}_\ell$
2. ESTIMATE: compute indicators $\eta_{u,\ell}(T)$, $\eta_{z,\ell}(T)$ for all $T \in \mathcal{T}_\ell$
3. find (minimal) set $\mathcal{M}_{u,\ell} \subseteq \mathcal{T}_\ell$ s.t. $\theta \sum_{T \in \mathcal{T}_\ell} \eta_{u,\ell}(T)^2 \leq \sum_{T \in \mathcal{M}_{u,\ell}} \eta_{u,\ell}(T)^2$
4. find (minimal) set $\mathcal{M}_{z,\ell} \subseteq \mathcal{T}_\ell$ s.t. $\theta \sum_{T \in \mathcal{T}_\ell} \eta_{z,\ell}(T)^2 \leq \sum_{T \in \mathcal{M}_{z,\ell}} \eta_{z,\ell}(T)^2$
5. MARK: choose $\mathcal{M}_\ell \in \{\mathcal{M}_{u,\ell}, \mathcal{M}_{z,\ell}\}$ with $\#\mathcal{M}_\ell = \min\{\#\mathcal{M}_{u,\ell}, \#\mathcal{M}_{z,\ell}\}$
6. REFINE: refine (at least) all $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$

Mommer, Stevenson: SINUM 47 (2009)

Optimal Convergence Rates

Theorem (Feischl, P., van der Zee '15+)

- validity of axioms (A1)–(A4) for $\eta_{u,\star}$ and $\eta_{z,\star}$
- "optimal mesh-refinement"
- $s, t > 0$ arbitrary
- $0 < \theta < \theta_{\rm opt} := (1 + C_{\rm stab}^2 C_{\rm rel}^2)^{-1}$
- $\mathcal{M}_{u,\ell}, \mathcal{M}_{z,\ell} \subseteq \mathcal{T}_\ell$ have (essentially) minimal cardinality

$$\Longrightarrow \quad \Big(\|\eta_u\|_{\mathbb{A}_s} + \|\eta_z\|_{\mathbb{A}_t} < \infty \quad \Longrightarrow \quad \eta_{u,\ell}\eta_{z,\ell} \lesssim (\#\mathcal{T}_\ell)^{-(s+t)} \Big)$$

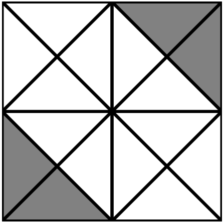
- generalizes earlier results beyond Poisson problem
- thorough analysis for algorithm from Becker et al.
- also applies to point errors in ABEM

Feischl, Führer, Gantner, Haberl, Praetorius: Numer. Math., online first '15

Feischl, Praetorius, van der Zee: Preprint arXiv #1505.04536

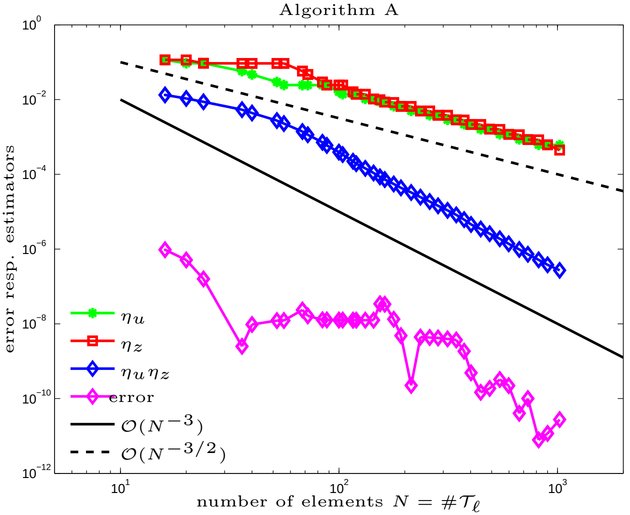
Numerical Example

- $-\Delta u = f$ in $\Omega = (0,1)^2$ with $u = 0$ on $\partial\Omega$
- $f(v) = -\int_{T_f} \partial_1 v \, dx$ right-hand side
- $q(u) = -\int_{T_q} \partial_1 u \, dx$ goal quantity



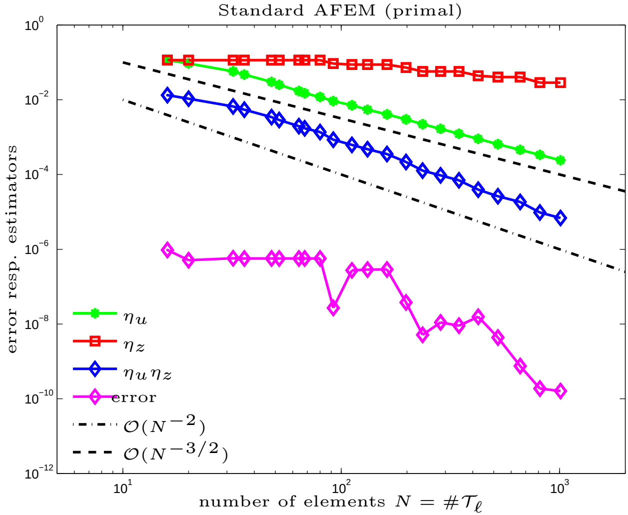
Mommer, Stevenson: SINUM 47 (2009)

Goal-Oriented Adaptivity, $\theta = 0.5$



- $U_\ell, Z_\ell \in \mathcal{S}^3_0(\mathcal{T}_\ell)$

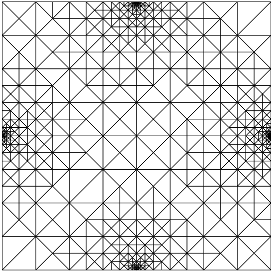
Standard Adaptivity, $\theta = 0.5$



- $U_\ell, Z_\ell \in \mathcal{S}_0^3(\mathcal{T}_\ell)$

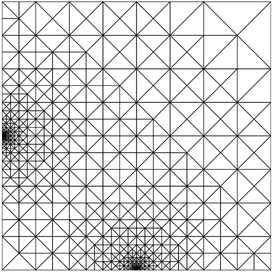
Adaptive Meshes

Algorithm A



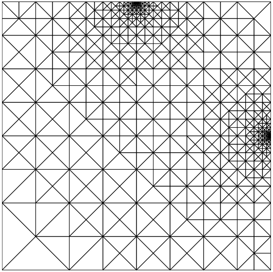
$\#\mathcal{T}_{38} = 1{,}022$

AFEM (primal)



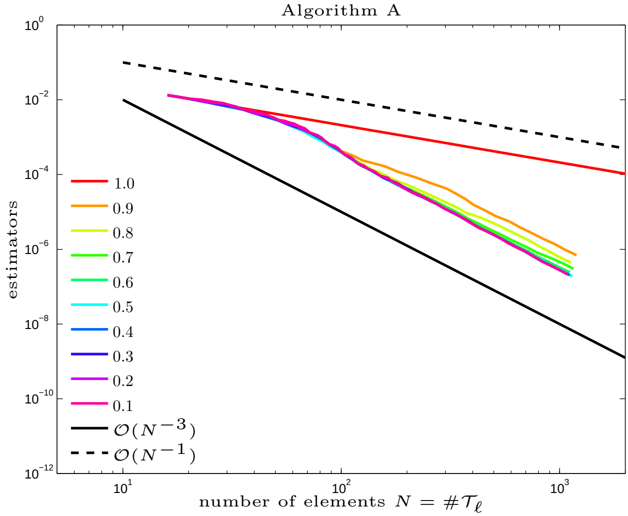
$\#\mathcal{T}_{22} = 1{,}010$

AFEM (dual)



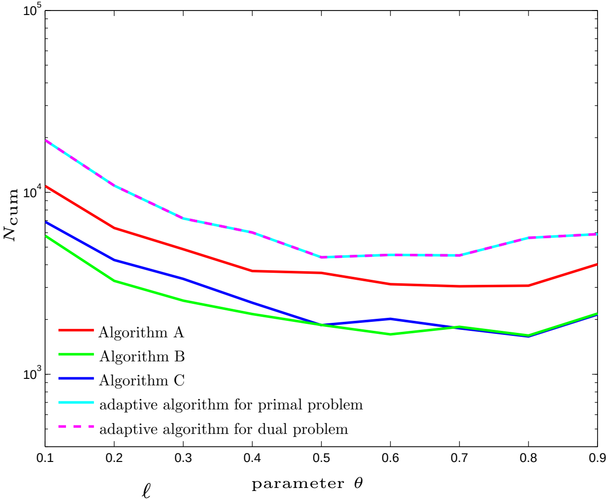
$\#\mathcal{T}_{22} = 1{,}010$

Goal-Oriented Adaptivity, $\theta = 0.1, \ldots, 1.0$



- $U_\ell, Z_\ell \in \mathcal{S}^3_0(\mathcal{T}_\ell)$

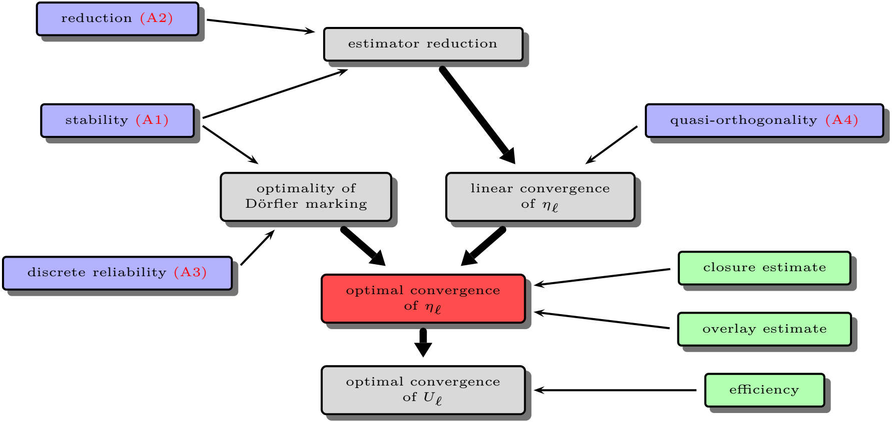
Computational Complexity



- consider: $N_{\rm cum} := \sum_{j=0}^{\ell} \#\mathcal{T}_j$ with $\eta_{u,\ell}\eta_{z,\ell} \le$ tol $= 10^{-5}$
- $U_\ell, Z_\ell \in \mathcal{S}_0^3(\mathcal{T}_\ell)$

Conclusions

Axioms of Adaptivity



Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Conclusions

- four axioms guarantee linear convergence with optimal rates
  - apply to adaptivity for energy error
  - apply to goal-oriented adaptivity
- independent of problem and discretization
- axioms are implicitly met in all results on rate optimality
- discrete reliability (A3) and quasi-orthogonality (A4) are sharp
- quasi-orth. (A4) is trivial for symmetric problems & Galerkin method
- axioms are valid for general 2nd order linear elliptic PDE
  - BEM with fixed polynomial degree $p \geq 1$ for symmetric integral operators
  - conforming FEM with fixed polynomial degree $p \geq 1$
  - inhomogeneous + mixed Dirichlet-Neumann-Robin BCs

Extensions

- theory extends to error estimators which violate (A1)–(A2)
  - e.g., ZZ-type error estimators
- main theorem holds if $\eta_\ell \sim \mu_\ell$ locally and μ_ℓ satisfies (A1)–(A4)
  - i.e., $\eta_\ell(T) \lesssim \mu_\ell\big(\mathrm{patch}(T)\big)$ and $\mu_\ell(T) \lesssim \eta_\ell\big(\mathrm{patch}(T)\big)$
- nonlinear energy minimization problems can be included
  - p-Laplace
  - strongly monotone operators
- proofs do not require norm, but only weak symmetry + triangle ineq.
  - consider $J(U_+) - J(U_\star)$ for energy minimization problems
- stopping criteria for iterative solvers can be included
  - requires (A1)–(A2) for arbitrary discrete functions

Thanks for Listening

- Carstensen, Feischl, Page, Praetorius: *Axioms of adaptivity*, Computers and Mathematics with Applications 67 (2014) (open access)
  - review of available results + general framework + ZZ + mixed BVP + ...
- Feischl, Praetorius, van der Zee: Preprint arXiv #1505.04536
  - general framework for optimal goal-oriented adaptivity + applications
- Feischl, Führer, Praetorius: SINUM 52 (2014)
  - FEM for 2nd order linear elliptic PDEs in $\mathbb{R}^d$ (possibly non-symmetric, quasi-linear)

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Mesh-Closure Estimate

Mesh-closure estimate

- arbitrary $\mathcal{M}_k \subseteq \mathcal{T}_k$
- successive refinement $\mathcal{T}_{k+1} = \texttt{refine}(\mathcal{T}_k, \mathcal{M}_k)$

$$\Longrightarrow \quad \#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C(\mathcal{T}_0) \sum_{j=0}^{\ell-1} \#\mathcal{M}_j$$

- newest vertex bisection
  - Binev, Dahmen, DeVore '04
  - Stevenson '08
  - Karkulik, Pavlicek, P. '13 (no assumption on $\mathcal{T}_0$ in 2D)
- red refinement with first-order hanging nodes
  - Bonito, Nochetto '10
- 2D red-green-blue refinement
  - Pavlicek, P. '11 (BSc thesis)

Overlay Estimate

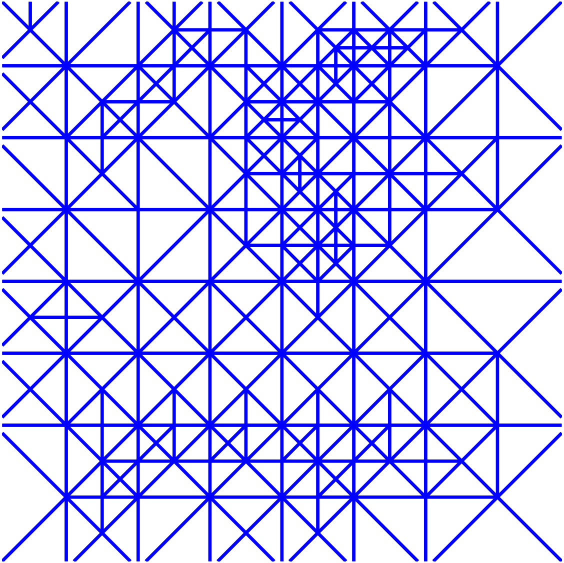
Overlay estimate

For $\mathcal{T}_\star, \mathcal{T}_\ell$ exists common refinement $\mathcal{T}_\star \oplus \mathcal{T}_\ell$ s.t.

$$\#(\mathcal{T}_\star \oplus \mathcal{T}_\ell) \leq \#\mathcal{T}_\star + \#\mathcal{T}_\ell - \#\mathcal{T}_0$$

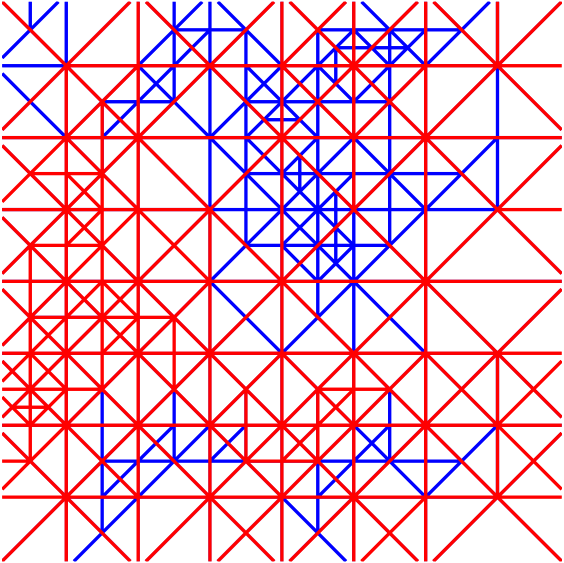
- newest vertex bisection
  - Stevenson '07
  - Cascón, Kreuzer, Nochetto, Siebert '08
- red refinement with first-order hanging nodes
  - Bonito, Nochetto '10
- wrong for 2D red-green-blue refinement
  - Pavlicek, P. '11 (BSc thesis)

Overlay Estimate: $\#(\mathcal{T}_\star \oplus \mathcal{T}_\ell) \le \#\mathcal{T}_\star + \#\mathcal{T}_\ell - \#\mathcal{T}_0$



- $\mathcal{T}_\star$ (blue)
- $\mathcal{T}_\ell$ (red)
- $\mathcal{T}_\star \oplus \mathcal{T}_\ell$

Overlay Estimate: $\#(\mathcal{T}_\star \oplus \mathcal{T}_\ell) \le \#\mathcal{T}_\star + \#\mathcal{T}_\ell - \#\mathcal{T}_0$



- $\mathcal{T}_\star$ (blue)
- $\mathcal{T}_\ell$ (red)
- $\mathcal{T}_\star \oplus \mathcal{T}_\ell$

Validity of Axioms

Theorem (Feischl, Führer, P. '14)

- $\boldsymbol{A} \in W^{1,\infty}$ symmetric, $\boldsymbol{b} \in L^\infty$, $c \in L^\infty$
- $\mathcal{L}u := -\nabla \cdot \boldsymbol{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu$
- induced bilinear form $a(\cdot,\cdot)$ is elliptic
- nestedness $\mathcal{S}^p_0(\mathcal{T}_\ell) \subset \mathcal{S}^p_0(\mathcal{T}_{\ell+1})$

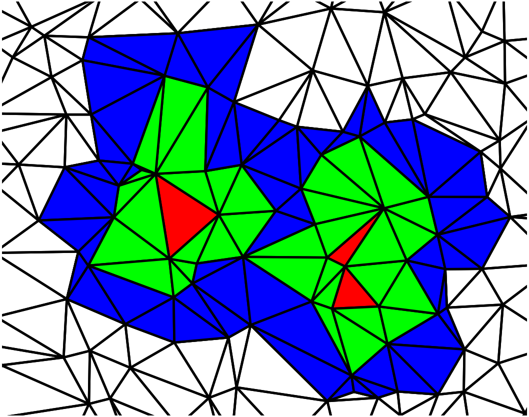
$\Longrightarrow$ weighted-residual error estimator satisfies (A1)–(A4)

Sketch of quasi-orthogonality (A3):

- $\mathcal{L}$ is compact perturbation of symmetric + elliptic operator
- $\|U_\infty - U_\ell\|_{H^1(\Omega)} \to 0$ for some $U_\infty \in H^1_0(\Omega)$
- moreover, $(U_\infty - U_\ell)/\|U_\infty - U_\ell\|_{H^1(\Omega)} \rightharpoonup 0$ weakly in $H^1(\Omega)$

Feischl, Führer, Praetorius: SINUM 52 (2014)

Patches



- $\mathcal{S} \subseteq \mathcal{T}_\ell$ set of elements
- 1st-order patch $\omega_\ell^1(\mathcal{S})$
- 2nd-order patch $\omega_\ell^2(\mathcal{S})$
- etc.

New Mesh-Size Function

- $\mathrm{diam}(T)$ does not necessarily shrink if T is refined
- remedy: consider $h_\ell|_T := |T|^{1/d}$
  1. $h_\ell|_T \simeq \mathrm{diam}(T)$
  2. $h_{\ell+1} \le h_\ell$ pointwise
  3. $h_{\ell+1} \le q\, h_\ell$ on refined elements $T \in \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}$
- sometimes, one needs reduction of h_ℓ on $\omega_\ell^k(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1})$

Proposition (Carstensen, Feischl, Page, P. 14)

- Fix arbitrary $k \in \mathbb{N}$
- For $\mathcal{T}_\ell$ exists mesh-size $h_\ell \in L^\infty(\Omega)$ s.t.
  1. $\mathrm{diam}(T) \lesssim h_\ell|_T \le \mathrm{diam}(T)$
  2. $h_{\ell+1} \le h_\ell$ pointwise
  3. $h_{\ell+1} \le q\, h_\ell$ on patch $\omega_\ell^k(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1})$