Aposteriori error analysis of timestepping schemes for the wave equation

using elliptic reconstruction techniques

Omar Lakkis

Mathematics — University of Sussex — Brighton, England UK

a talk based on joint work with

E.H. Georgoulis (Athens GR & Leicester GB), C. Makridakis (Sussex GB & Crete GR), J.M. Virtanen (Leicester GB)

6 January 2016

Adaptive algorithms for computational partial differential equations

University of Birmingham

England UK

The model linear wave equation

a second order hyperbolic problem

Initial–boundary value problem

$$\partial_{tt} u = \Delta u + f \quad , u|_{\text{spatial boundary}} = 0 \text{ and } u(0) = u_0,\ \partial_t u(0) = v_0.$$

The model linear wave equation

a second order hyperbolic problem

Initial–boundary value problem

$$\partial_{tt} u = \Delta u + f \quad , u|_{\text{spatial boundary}} = 0 \text{ and } u(0) = u_0,\ \partial_t u(0) = v_0.$$

As a first order system

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

The model linear wave equation

a second order hyperbolic problem

Initial–boundary value problem

$$\partial_{tt} u = \Delta u + f \quad , u|_{\text{spatial boundary}} = 0 \text{ and } u(0) = u_0,\ \partial_t u(0) = v_0.$$

As a first order system

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

Simplest timestepping scheme: Backward–Euler for the system (Bernardi and Süli, 2005):

$$\begin{bmatrix} 1 & -k_n \\ +k_n \Delta & 1 \end{bmatrix} \begin{bmatrix} u^n \\ v^n \end{bmatrix} = \begin{bmatrix} u^{n-1} \\ v^{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ f(t_n) \end{bmatrix}.$$

The model linear wave equation

a second order hyperbolic problem

Initial–boundary value problem

$$\partial_{tt} u = \Delta u + f \quad , u|_{\text{spatial boundary}} = 0 \text{ and } u(0) = u_0, \; \partial_t u(0) = v_0.$$

As a first order system

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

Simplest timestepping scheme: Backward–Euler for the system (Bernardi and Süli, 2005):

$$\begin{bmatrix} 1 & -k_n \\ +k_n \Delta & 1 \end{bmatrix} \begin{bmatrix} u^n \\ v^n \end{bmatrix} = \begin{bmatrix} u^{n-1} \\ v^{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ f(t_n) \end{bmatrix}.$$

Resulting equation form is a 2-step (timestep $= k_n$) method

$$\frac{u^n - u^{n-1}}{k_n} - \frac{u^{n-1} - u^{n-2}}{k_{n-1}} - k_n \Delta u^n = k_n f(t_n).$$

Previous work

Goal-oriented duality approach:
W. Bangerth and R. Rannacher (1999) and
Wolfgang Bangerth and Rolf Rannacher (2001)

Direct Galerkin orthogonality with energy approach: Bernardi and Süli (2005)

Semidiscrete analysis:
Picasso (2010)

Heuristic-based adaptive methods:
Wiberg & Li (1998), Schemann & Bornemann (1998), Romero & Lacoma (2006).

Spatially semidiscrete schemes

Suppose the exact elliptic operator $\mathscr{A} : \operatorname{Dom} \mathscr{A} \to \operatorname{Ran} \mathscr{A}$, e.g, $-\Delta : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is discretized as

$$
(8.1) \qquad \begin{array}{rccl} A : & \mathbb{V} & \to & \mathbb{V} \\ & V & \mapsto & AV : \langle AV, \Phi \rangle = \langle \mathscr{A} V \mid \Phi \rangle \quad \forall\, \Phi \in \mathbb{V} \end{array}
$$

then a spatially semidiscrete method for the wave equation takes the system form

$$
\mathrm{d}_t \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ A & 0 \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \Pi^{\mathbb{V}} f(t) \end{bmatrix}
$$

where $\Pi^{\mathbb{V}}$ is the $\mathrm{L}_2(\Omega)$-orthogonal projection.

Spatially semidiscrete schemes

Suppose the exact elliptic operator $\mathscr{A} : \operatorname{Dom}\mathscr{A} \to \operatorname{Ran}\mathscr{A}$, e.g, $-\Delta : \mathrm{H}_0^1(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is discretized as

$$\begin{aligned} A : \quad & \mathbb{V} \quad \to \quad \mathbb{V} \\ & V \quad \mapsto \quad AV : \langle AV, \Phi\rangle = \langle \mathscr{A} V \,|\, \Phi\rangle \quad \forall\, \Phi \in \mathbb{V} \end{aligned} \tag{9.1}$$

then a spatially semidiscrete method for the wave equation takes the system form

$$\mathrm{d}_t \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ A & 0 \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \Pi^{\mathbb{V}} f(t) \end{bmatrix}$$

where $\Pi^{\mathbb{V}}$ is the $\mathrm{L}_2(\Omega)$-orthogonal projection.
Equation form looks very nice (it is) to analyze

$$\mathrm{d}_{tt}\, U(t) + AU(t) = \Pi^{\mathbb{V}} f(t).$$

Energy methods for the wave equation

Combining the elliptic reconstruction to energy estimates for the system, one recovers the estimates of Bernardi and Süli (2005).

Energy methods for the wave equation

Combining the elliptic reconstruction to energy estimates for the system, one recovers the estimates of Bernardi and Süli (2005). Combining the elliptic reconstruction to energy estimates for the equation form, one recovers more general estimates, but only in the spatially discrete scheme.

Combining the elliptic reconstruction to energy estimates for the system, one recovers the estimates of Bernardi and Süli (2005). Combining the elliptic reconstruction to energy estimates for the equation form, one recovers more general estimates, but only in the spatially discrete scheme.

Remark (Failure of fully discrete analysis in energy for equation)

is due to "bad" time reconstruction, using polynomials, and there seems to be no way out...

Combining the elliptic reconstruction to energy estimates for the system, one recovers the estimates of Bernardi and Süli (2005). Combining the elliptic reconstruction to energy estimates for the equation form, one recovers more general estimates, but only in the spatially discrete scheme.

Remark (Failure of fully discrete analysis in energy for equation)

is due to “bad” time reconstruction, using polynomials, and there seems to be no way out...

Fully discrete scheme

We consider the fully discrete scheme for the initial value wave problem

$$\text{for each } n = 1, \ldots, N, \text{ find } U^n \in V_h^n \text{ such that}$$
$$\langle \partial^2 U^n, V \rangle + a(U^n, V) = \langle f^n, V \rangle \quad \forall V \in V_h^n,$$

where $f^n := f(t^n, \cdot)$, the backward second and first finite differences

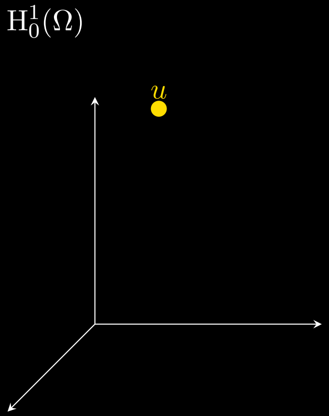
$$\partial^2 U^n := \frac{\partial U^n - \partial U^{n-1}}{k_n},$$

with

$$\partial U^n := \begin{cases} \dfrac{U^n - U^{n-1}}{k_n}, & \text{for } n = 1, 2, \ldots, N, \\ V^0 := \pi^0 u_1 & \text{for } n = 0, \end{cases}$$

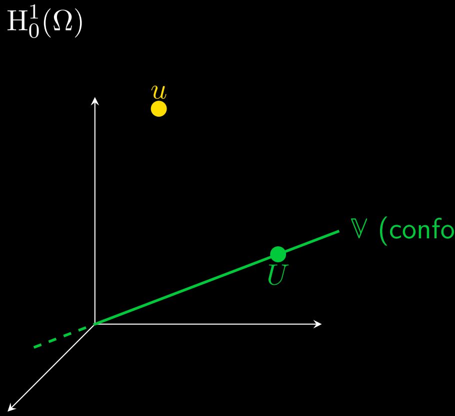
A User's Guide to the Elliptic Reconstruction

- $u(\cdot, t) \in \mathrm{H}_0^1(\Omega)$ exact solution at fixed time t



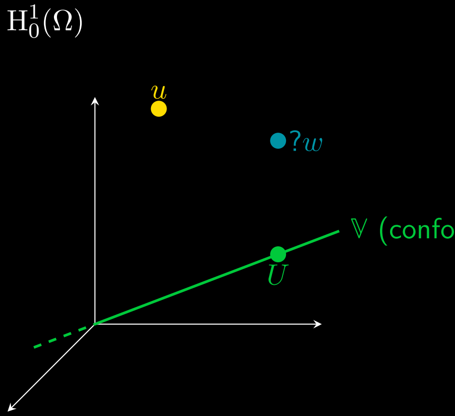
A User's Guide to the Elliptic Reconstruction

- $u(\cdot,t) \in \mathrm{H}^1_0(\Omega)$ exact solution at fixed time t
- $U(\cdot,t) \in \mathbb{V}$ time-dependent $\mathbb{V}$-FE approximation of $u(\cdot,t)$.



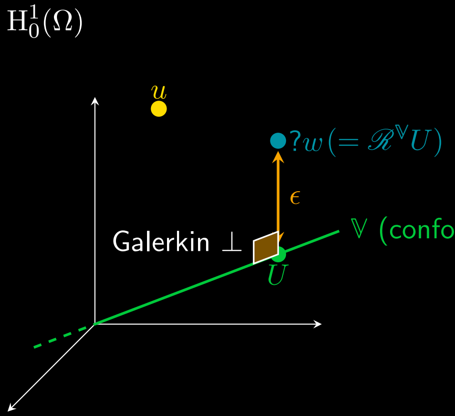
A User's Guide to the Elliptic Reconstruction

- $u(\cdot,t) \in \mathrm{H}^1_0(\Omega)$ exact solution at fixed time t
- $U(\cdot,t) \in \mathbb{V}$ time-dependent $\mathbb{V}$-FE approximation of $u(\cdot,t)$.
- want: w intermediate object between u and U

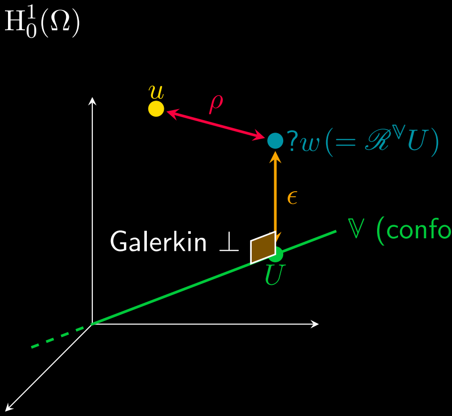


A User's Guide to the Elliptic Reconstruction

- $u(\cdot, t) \in \mathrm{H}_0^1(\Omega)$ exact solution at fixed time t
- $U(\cdot, t) \in \mathbb{V}$ time-dependent $\mathbb{V}$-FE approximation of $u(\cdot, t)$.
- want: w intermediate object between u and U
- want: $U \in \mathbb{V}$ an elliptic $\mathbb{V}$-FE approximation of w, error $\epsilon = U - w$,

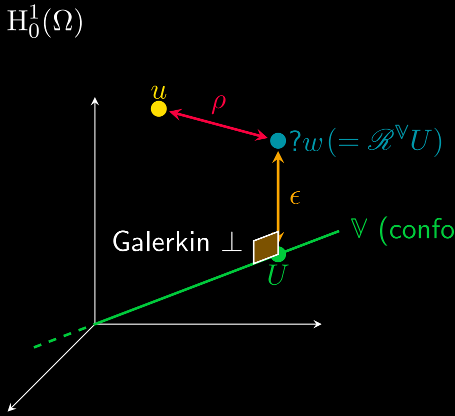


- $u(\cdot,t) \in \mathrm{H}_0^1(\Omega)$ exact solution at fixed time t
- $U(\cdot,t) \in \mathbb{V}$ time-dependent $\mathbb{V}$-FE approximation of $u(\cdot,t)$.
- want: w intermediate object between u and U
- want: $U \in \mathbb{V}$ an elliptic $\mathbb{V}$-FE approximation of w, error $\epsilon = U - w$,
- Galerkin orthogonality $\Rightarrow$ aposteriori bounds on $\|\epsilon\|$ are available "off the shelf",



A User's Guide to the Elliptic Reconstruction

- $u(\cdot,t) \in \mathrm{H}^1_0(\Omega)$ exact solution at fixed time t
- $U(\cdot,t) \in \mathbb{V}$ time-dependent $\mathbb{V}$-FE approximation of $u(\cdot,t)$.
- want: w intermediate object between u and U
- want: $U \in \mathbb{V}$ an elliptic $\mathbb{V}$-FE approximation of w, error $\epsilon = U - w$,
- Galerkin orthogonality $\Rightarrow$ aposteriori bounds on $\|\epsilon\|$ are available "off the shelf",
- w not computable,
- but time dependent error $\rho := w - u$ satifies original PDE with computable data.



- Combining the original wave equation

$$\partial_{tt} u + \mathscr{A} u = f \tag{21.1}$$

- Combining the original wave equation

$$\partial_{tt} u + \mathscr{A} u = f \tag{22.1}$$

- semidiscrete scheme

$$\partial_{tt} U + A U = \Pi^{\mathbb{V}} f \tag{22.2}$$

- Combining the original wave equation

$$\partial_{tt} u + \mathscr{A} u = f \tag{23.1}$$

- semidiscrete scheme

$$\partial_{tt} U + A U = \Pi^{\mathbb{V}} f \tag{23.2}$$

- elliptic reconstruction $w := \mathscr{R}^{\mathbb{V}} U \in \mathrm{H}^1_0(\Omega)$ such that

$$\mathscr{A} \mathscr{R}^{\mathbb{V}} U = A U \tag{23.3}$$

- Combining the original wave equation

$$\partial_{tt} u + \mathscr{A} u = f \tag{24.1}$$

- semidiscrete scheme

$$\partial_{tt} U + AU = \Pi^{\mathbb{V}} f \tag{24.2}$$

- elliptic reconstruction $w := \mathscr{R}^{\mathbb{V}} U \in \mathrm{H}_0^1(\Omega)$ such that

$$\mathscr{A} \mathscr{R}^{\mathbb{V}} U = AU \tag{24.3}$$

- error splitting $e := U - u = U - w + w - u =: -\epsilon + \rho$

- Combining the original wave equation

$$\partial_{tt} u + \mathscr{A} u = f \tag{25.1}$$

- semidiscrete scheme

$$\partial_{tt} U + A U = \Pi^{\mathbb{V}} f \tag{25.2}$$

- elliptic reconstruction $w := \mathscr{R}^{\mathbb{V}} U \in \mathrm{H}^1_0(\Omega)$ such that

$$\mathscr{A} \mathscr{R}^{\mathbb{V}} U = A U \tag{25.3}$$

- error splitting $e := U - u = U - w + w - u =: -\epsilon + \rho$
-

$$\partial_{tt}[w - u] + \mathscr{A}\,[w - u] = \partial_{tt}[w - U] + \left[\Pi^{\mathbb{V}} - \mathrm{id}\right] f \tag{25.4}$$

Elliptic reconstruction: a user's guide (**controlling ρ**)

$\partial_{tt}\rho + \mathscr{A}\rho = \partial_{tt}\epsilon + \mathscr{A}(w - w^n) +$ controlled terms

control of the spatial error $\partial_{tt}\epsilon$:

Elliptic reconstruction: a user's guide (**controlling** ρ)

$\partial_{tt}\rho + \mathscr{A}\rho = \partial_{tt}\epsilon + \mathscr{A}(w - w^n) +$ controlled terms

control of the spatial error $\partial_{tt}\epsilon$:

- Use PDE for ρ with $\partial_{tt}\epsilon$ as data to obtain bound on $\|\rho\| < C\,\|\partial_{tt}\epsilon\|$.

Elliptic reconstruction: a user's guide (**controlling ρ**)

$\partial_{tt}\rho + \mathscr{A}\rho = \partial_{tt}\epsilon + \mathscr{A}(w - w^n) +$ controlled terms

control of the spatial error $\partial_{tt}\epsilon$:

- Use PDE for ρ with $\partial_{tt}\epsilon$ as data to obtain bound on $\|\rho\| < C\,\|\partial_{tt}\epsilon\|$.
- $\partial_{tt}\epsilon = \partial_{tt}w - \partial_{tt}U = \mathscr{R}\partial_{tt}U - \partial_{tt}U$.

Elliptic reconstruction: a user's guide (**controlling ρ**)

$\partial_{tt}\rho + \mathscr{A}\rho = \partial_{tt}\epsilon + \mathscr{A}(w - w^n) +$ controlled terms

control of the spatial error $\partial_{tt}\epsilon$:

- Use PDE for ρ with $\partial_{tt}\epsilon$ as data to obtain bound on $\|\rho\| < C\,\|\partial_{tt}\epsilon\|$.
- $\partial_{tt}\epsilon = \partial_{tt}w - \partial_{tt}U = \mathscr{R}\partial_{tt}U - \partial_{tt}U$.
- $\partial_{tt}U$ elliptic $\mathbb{V}_h$-FE solution with exact $\mathscr{R}\partial_{tt}U = \partial_{tt}w \in \mathrm{H}^1_0(\Omega)$.

Elliptic reconstruction: a user's guide (controlling ρ)

$\partial_{tt}\rho + \mathscr{A}\rho = \partial_{tt}\epsilon + \mathscr{A}(w - w^n)$ + controlled terms

control of the spatial error $\partial_{tt}\epsilon$:

- Use PDE for ρ with $\partial_{tt}\epsilon$ as data to obtain bound on $\|\rho\| < C\,\|\partial_{tt}\epsilon\|$.
- $\partial_{tt}\epsilon = \partial_{tt}w - \partial_{tt}U = \mathscr{R}\partial_{tt}U - \partial_{tt}U$.
- $\partial_{tt}U$ elliptic $\mathbb{V}_h$-FE solution with exact $\mathscr{R}\partial_{tt}U = \partial_{tt}w \in \mathrm{H}_0^1(\Omega)$.
- $\Rightarrow\ \|\partial_{tt}\epsilon\|$ elliptic error controlled aposteriori by estimator $\mathscr{E}[\partial_{tt}U, \partial_{tt}f, \mathbb{V}_h]$.

Elliptic reconstruction: a user's guide (**controlling** ρ)

$\partial_{tt}\rho + \mathscr{A}\rho = \partial_{tt}\epsilon + \mathscr{A}(w - w^n) +$ controlled terms

control of the spatial error $\partial_{tt}\epsilon$:

- Use PDE for ρ with $\partial_{tt}\epsilon$ as data to obtain bound on $\|\rho\| < C\|\partial_{tt}\epsilon\|$.
- $\partial_{tt}\epsilon = \partial_{tt}w - \partial_{tt}U = \mathscr{R}\partial_{tt}U - \partial_{tt}U$.
- $\partial_{tt}U$ elliptic $\mathbb{V}_h$-FE solution with exact $\mathscr{R}\partial_{tt}U = \partial_{tt}w \in \mathrm{H}_0^1(\Omega)$.
- $\Rightarrow$ $\|\partial_{tt}\epsilon\|$ elliptic error controlled aposteriori by estimator $\mathscr{E}[\partial_{tt}U, \partial_{tt}f, \mathbb{V}_h]$.
- Hence $\|\rho\| \leq C\mathscr{E}[\partial_{tt}U, \partial_{tt}f, \mathbb{V}_h]$

control of the time error $\mathscr{A}(w - w^n)$:

Elliptic reconstruction: a user's guide (**controlling** ρ)

$\partial_{tt}\rho + \mathscr{A}\rho = \partial_{tt}\epsilon + \mathscr{A}(w - w^n) +$ controlled terms

control of the spatial error $\partial_{tt}\epsilon$:

- Use PDE for ρ with $\partial_{tt}\epsilon$ as data to obtain bound on $\|\rho\| < C\,\|\partial_{tt}\epsilon\|$.
- $\partial_{tt}\epsilon = \partial_{tt}w - \partial_{tt}U = \mathscr{R}\partial_{tt}U - \partial_{tt}U$.
- $\partial_{tt}U$ elliptic $\mathbb{V}_h$-FE solution with exact $\mathscr{R}\partial_{tt}U = \partial_{tt}w \in \mathrm{H}^1_0(\Omega)$.
- $\Rightarrow$ $\|\partial_{tt}\epsilon\|$ elliptic error controlled aposteriori by estimator $\mathscr{E}[\partial_{tt}U, \partial_{tt}f, \mathbb{V}_h]$.
- Hence $\|\rho\| \leq C\mathscr{E}[\partial_{tt}U, \partial_{tt}f, \mathbb{V}_h]$

control of the time error $\mathscr{A}(w - w^n)$:choice depending on PDE book used.

Elliptic reconstruction: a user's guide (**controlling** ρ)

$\partial_{tt}\rho + \mathscr{A}\rho = \partial_{tt}\epsilon + \mathscr{A}(w - w^n) +$ controlled terms

control of the spatial error $\partial_{tt}\epsilon$:

- Use PDE for ρ with $\partial_{tt}\epsilon$ as data to obtain bound on $\|\rho\| < C\,\|\partial_{tt}\epsilon\|$.
- $\partial_{tt}\epsilon = \partial_{tt}w - \partial_{tt}U = \mathscr{R}\partial_{tt}U - \partial_{tt}U$.
- $\partial_{tt}U$ elliptic $\mathbb{V}_h$-FE solution with exact $\mathscr{R}\partial_{tt}U = \partial_{tt}w \in \mathrm{H}_0^1(\Omega)$.
- $\Rightarrow$ $\|\partial_{tt}\epsilon\|$ elliptic error controlled aposteriori by estimator $\mathscr{E}[\partial_{tt}U, \partial_{tt}f, \mathbb{V}_h]$.
- Hence $\|\rho\| \leq C\mathscr{E}[\partial_{tt}U, \partial_{tt}f, \mathbb{V}_h]$

control of the time error $\mathscr{A}(w - w^n)$:choice depending on PDE book used.

Example: use relation

$$\partial_{tt}U + \mathscr{A}w^n = \Pi^n f^n$$

leads to explicit aposteriori representation

$$\mathscr{A}w^n := \Pi^n f^n - \partial_{tt}U.$$

A Baker's recipe

introduced by Baker (1976) for $\mathrm{L}_\infty(0,T;\mathrm{L}_2(\Omega))$ wave equation apriori error estimates

A Baker's recipe

introduced by Baker (1976) for $\mathrm{L}_\infty(0,T;\mathrm{L}_2(\Omega))$ wave equation apriori error estimates

Theorem (abstract semidiscrete error bound by Georgoulis, Lakkis and Makridakis, 2013)

Let $u(t)$ exact solution, $U(t)$ space-discrete, $\mathscr{R}u(t)$ elliptic reconstruction, and $\epsilon = u - U$. Then

(35.1)

$$\|e\|_{L^\infty(0,T;L^2(\Omega))} \leq \|\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{2}\Big(\|u_0 - U(0)\| + \|\epsilon(0)\|\Big) + 2\int_0^T \|\epsilon_t\| + C_{a,T}\|u_1 - U_t(0)\|,$$

where $C_{a,T} := \min\{2T, \sqrt{2C_\Omega/\alpha_{\min}}\}$, where C_Ω is the constant of the Poincaré–Friedrichs inequality $\|v\|^2 \leq C_\Omega\|\nabla v\|^2$, for $v \in H_0^1(\Omega)$.

Proof: test with $v(\cdot,t) = \int_\tau^t \rho(\cdot,s)\,\mathrm{d}\,s$ for each τ and then take $\max_\tau$.

Work plan

1. From semidiscrete case we need:

Work plan

1. From semidiscrete case we need:
   a. elliptic reconstruction $w(\cdot, t) := \mathscr{R}^{\mathbb{V}} U(\cdot, t)$ to write an error relation

Work plan

1. From semidiscrete case we need:
   a. elliptic reconstruction $w(\cdot,t) := \mathscr{R}^{\mathbb{V}} U(\cdot,t)$ to write an error relation
   b. Baker's function $v(\cdot,t) := \int_0^t w(\cdot,s) - u(\cdot,s) \,\mathrm{d}\, s$

Work plan

1. From semidiscrete case we need:
   - a. elliptic reconstruction $w(\cdot,t) := \mathscr{R}^{\mathbb{V}} U(\cdot,t)$ to write an error relation
   - b. Baker's function $v(\cdot,t) := \int_0^t w(\cdot,s) - u(\cdot,s) \,\mathrm{d}\, s$
2. To analyze the fully discrete scheme we also need:

Work plan

1. From semidiscrete case we need:
   a. elliptic reconstruction $w(\cdot,t) := \mathscr{R}^{\mathbb{V}} U(\cdot,t)$ to write an error relation
   b. Baker's function $v(\cdot,t) := \int_0^t w(\cdot,s) - u(\cdot,s) \,\mathrm{d}\, s$
2. To analyze the fully discrete scheme we also need:
   a. Discrete Baker's test function.

Work plan

1. From semidiscrete case we need:
   - a. elliptic reconstruction $w(\cdot,t) := \mathscr{R}^{\mathbb{V}} U(\cdot,t)$ to write an error relation
   - b. Baker's function $v(\cdot,t) := \int_0^t w(\cdot,s) - u(\cdot,s)\,\mathrm{d}\,s$
2. To analyze the fully discrete scheme we also need:
   - a. Discrete Baker's test function.
   - b. a special cubic time-reconstruction satisfying a crucial vanishing moment property

Theorem (abstract fully-discrete error bound by Georgoulis, Lakkis and Makridakis, 2013)

Let w^n be the elliptic reconstruction of U^n, $n = 0, \ldots, N$, and consider the $\mathrm{C}^{1,1}$-piecewise quadratic extension w, U, of both functions respectively,

(42.1)
$$U(t) := \frac{t - t^{n-1}}{k_n} U^n + \frac{t^n - t}{k_n} U^{n-1} - \frac{(t - t^{n-1})(t^n - t)^2}{k_n} \partial^2 U^n \text{ and } \partial^2 U^n := \frac{\partial U^n - \partial U^{n-1}}{k_n}$$
$$\epsilon := w - u \text{ and } e := u - U$$

where w is the elliptic reconstruction of u and $\eta_1(\tau), \ldots, \eta_4(\tau)$ be appropriate (time τ-dependent) error indicators then

(42.2)
$$\begin{aligned} \|e\|_{L^\infty(0,t^N;L^2(\Omega))} \leq & \|\epsilon\|_{L^\infty(0,t^N;L^2(\Omega))} + \sqrt{2}\Big(\|u_0 - U(0)\| + \|\epsilon(0)\|\Big) \\ & + 2\Big(\int_0^{t^N} \|\epsilon_t\| + \sum_{i=1}^{4} \eta_i(t^N)\Big) + C_{a,N}\|u_1 - V^0\|, \end{aligned}$$

where $C_{a,N} := \min\{2t^N, \sqrt{2C_\Omega/\alpha_{\min}}\}$, C_Ω is Poincaré–Friedrichs inequality constant.

Error indicators I

For any $\tau \in \left(t^{m-1}, t^m\right]$

mesh change indicator

$\eta_1(\tau) := \eta_{1,1}(\tau) + \eta_{1,2}(\tau)$, with

$$\eta_{1,1}(\tau) := \sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} \|(I - \Pi^j)U_t\| + \int_{t^{m-1}}^{\tau} \|(I - \Pi^m)U_t\|,$$

$$\eta_{1,2}(\tau) := \sum_{j=1}^{m-1} (\tau - t^j)\|(\Pi^{j+1} - \Pi^j)\partial U^j\| + \tau\|(I - \Pi^0)V^0(0)\|,$$

Error indicators II

evolution error indicator

$$\eta_2(\tau) := \int_0^\tau \|\mathcal{G}\|,$$

where $\mathcal{G} : (0,T] \to \mathbb{R}$ with $\mathcal{G}|_{(t^{j-1},t^j]} := \mathcal{G}^j$, $j = 1,\dots,N$ and

$$\mathcal{G}^j(t) := \frac{(t^j - t)^2}{2}\partial g^j - \Big(\frac{(t^j - t)^4}{4k_j} - \frac{(t^j - t)^3}{3}\Big)\partial^2 g^j - \gamma_j,$$

with

$$g^n := A^n U^n - \Pi^n f^n + \bar{f}^n,$$

$$\gamma_j := \begin{cases} 0 & \text{for } j = 0 \\ \gamma_{j-1} + \frac{k_j^2}{2}\partial g^j + \frac{k_j^3}{12}\partial^2 g^j & \text{if } j = 1,\dots,N. \end{cases}$$

Error indicators III

data error indicator

$$\eta_3(\tau) := \frac{1}{2\pi} \sum_{j=1}^{m-1} \left(\int_{t^{j-1}}^{t^j} k_j^3 \|\bar{f}^j - f\|^2 \right)^{1/2} + \left(\int_{t^{m-1}}^{\tau} k_m^3 \|\bar{f}^m - f\|^2 \right)^{1/2};$$

time reconstruction error indicator

$$\eta_4(\tau) := \frac{1}{2\pi} \sum_{j=1}^{m-1} \left(\int_{t^{j-1}}^{t^j} k_j^3 \|\mu^j \partial^2 U^j\|^2 \right)^{1/2} + \left(\int_{t^{m-1}}^{\tau} k_m^3 \|\mu^m \partial^2 U^m\|^2 \right)^{1/2}.$$

What is μ^n? I

Proposition (fully-discrete error relation)

Denote $\rho := u - w$ $(w := \mathscr{R}u)$ *and* $t \in (t^{n-1}, t^n]$, $n = 1, \dots, N$, *we have*

$$\langle e_{tt}, v\rangle + a(\rho, v) = \langle (I - \Pi^n) U_{tt}, v\rangle + \mu^n \langle \partial^2 U^n, \Pi^n v\rangle + a(w - w^n, v) + \langle \bar{f}^n - f, v\rangle \tag{46.1}$$

for all $v \in H^1_0(\Omega)$, *with* $\Pi^n : L^2(\Omega) \to V^n_h$ *denoting the orthogonal* L^2*-projection operator onto* V^n_h, I *is the identity mapping in* $L^2(\Omega)$, *and*

$$\mu^n(t) := -6k_n^{-1}(t - t^{n-\frac{1}{2}}),$$

where $t^{n-\frac{1}{2}} := \frac{1}{2}(t^n + t^{n-1})$.

What is μ^n? II

Remark (vanishing moments)

Crucially the functions μ^n satisfy:

$$\int_{t^{n-1}}^{t^n} \mu^n(t)\,\mathrm{d}\,t = 0.$$

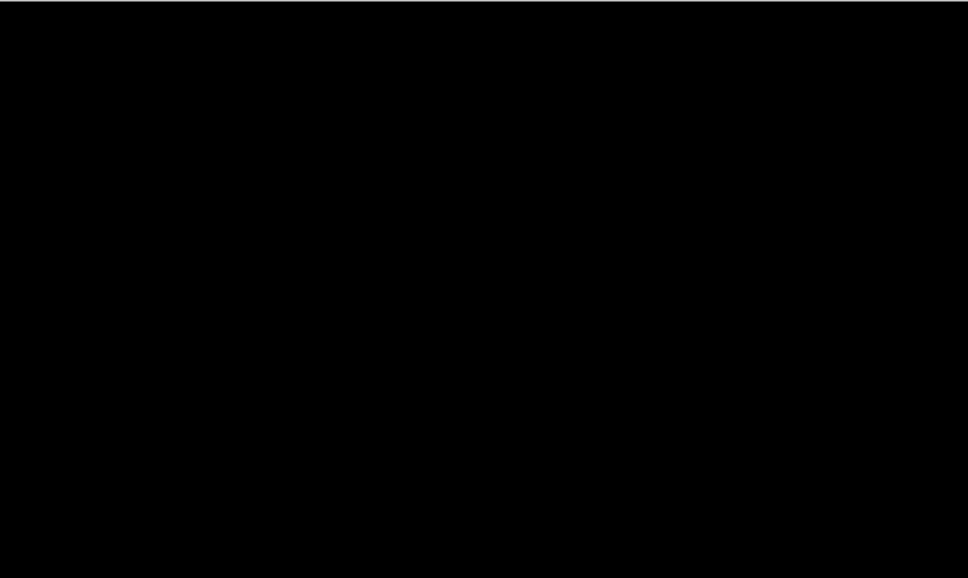
so when integrating in time

$$\text{(46.2)}\qquad \begin{aligned}\langle e_{tt}, v\rangle + a(\rho, v) =& \langle (I - \Pi^n)U_{tt}, v\rangle + \mu^n \langle \partial^2 U^n, \Pi^n v\rangle \\ & + a(w - w^n, v) + \langle \bar{f}^n - f, v\rangle\end{aligned}$$

the μ^n terms disappear leading to optimal order error indicators.

Verlet, leapfrog and cosine methods

analysis is useful for useful methods Georgoulis, Lakkis, Makridakis and Virtanen, 2014



Verlet, leapfrog and cosine methods

analysis is useful for useful methods Georgoulis, Lakkis, Makridakis and Virtanen, 2014

- Main drawback: backward Euler is too dissipative to be useful.

Verlet, leapfrog and cosine methods

analysis is useful for useful methods Georgoulis, Lakkis, Makridakis and Virtanen, 2014

- Main drawback: backward Euler is too dissipative to be useful.
- In practice one must employ conservative methods, and conserve stability.

Verlet, leapfrog and cosine methods

analysis is useful for useful methods Georgoulis, Lakkis, Makridakis and Virtanen, 2014

- Main drawback: backward Euler is too dissipative to be useful.
- In practice one must employ conservative methods, and conserve stability.
- Conservation is harder for hyperbolic equations than parabolic.

Verlet, leapfrog and cosine methods

analysis is useful for useful methods Georgoulis, Lakkis, Makridakis and Virtanen, 2014

- Main drawback: backward Euler is too dissipative to be useful.
- In practice one must employ conservative methods, and conserve stability.
- Conservation is harder for hyperbolic equations than parabolic.
- Stability is easier for explicit schemes: "$\Delta x = \Delta t$".

Verlet, leapfrog and cosine methods

analysis is useful for useful methods Georgoulis, Lakkis, Makridakis and Virtanen, 2014

- Main drawback: backward Euler is too dissipative to be useful.
- In practice one must employ conservative methods, and conserve stability.
- Conservation is harder for hyperbolic equations than parabolic.
- Stability is easier for explicit schemes: "$\Delta x = \Delta t$".
- Popular method coming from mechanics is Verlet's method used for rigid motion in astronomy.

- Main drawback: backward Euler is too dissipative to be useful.
- In practice one must employ conservative methods, and conserve stability.
- Conservation is harder for hyperbolic equations than parabolic.
- Stability is easier for explicit schemes: "$\Delta x = \Delta t$".
- Popular method coming from mechanics is Verlet's method used for rigid motion in astronomy.
- Generalization to wave is known as leapfrog method.

Verlet, leapfrog and cosine methods

analysis is useful for useful methods Georgoulis, Lakkis, Makridakis and Virtanen, 2014

- Main drawback: backward Euler is too dissipative to be useful.
- In practice one must employ conservative methods, and conserve stability.
- Conservation is harder for hyperbolic equations than parabolic.
- Stability is easier for explicit schemes: "$\Delta x = \Delta t$".
- Popular method coming from mechanics is Verlet's method used for rigid motion in astronomy.
- Generalization to wave is known as leapfrog method.
- Further generalizable to family of cosine methods, in similar spirit to exponential methods for first order DE's.

Verlet's method

- First recorded use by Newton (1687),

Verlet's method

- First recorded use by Newton (1687),
- Delambre (1791),

Verlet's method

- First recorded use by Newton (1687),
- Delambre (1791),
- Cowell and Crommelin (1909) in comet-orbit,

Verlet's method

- First recorded use by Newton (1687),
- Delambre (1791),
- Cowell and Crommelin (1909) in comet-orbit,
- Størmer (1907) in electromagnetics.

Verlet's method

- First recorded use by Newton (1687),
- Delambre (1791),
- Cowell and Crommelin (1909) in comet-orbit,
- Størmer (1907) in electromagnetics.
- Also known as the Newmark schemes.

Verlet's method

- First recorded use by Newton (1687),
- Delambre (1791),
- Cowell and Crommelin (1909) in comet-orbit,
- Størmer (1907) in electromagnetics.
- Also known as the Newmark schemes.
- Rediscovered and popularized amongst modern physicists (for molecular dynamics) with modern computers in 1960's by Loup Verlet as a two-step method:
  (61.1)

$$\ddot{u} = A(u),\ u(0) = u_0,\ \dot{u}(0) = v_0 \Leftrightarrow \begin{cases} \dot{u} = v & u(0) = u_0 \\ \dot{v} = A(u) & v(0) = v_0 \end{cases}$$

$$u^1 := u_0 + v_0 k + \frac{1}{2} A(u_0)\,(k)^2$$

$$u^{n+1} k = 2u^n - u^{n-1} + A(u^n)\,(k)^2$$

The "simplest" Verlet method

$$\ddot{u} = A(u),\ u(0) = u_0,\ \dot{u}(0) = v_0 \Leftrightarrow \begin{cases} \dot{u} = v & u(0) = u_0 \\ \dot{v} = A(u) & v(0) = v_0 \end{cases}$$

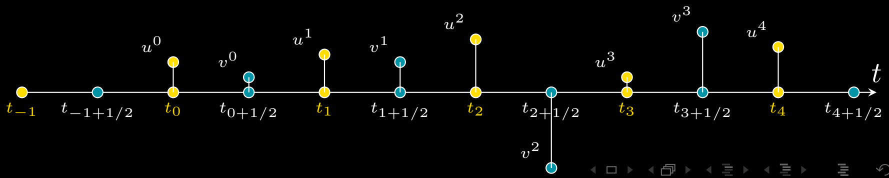
(62.1)

$$u^1 := u_0 + v_0 k + \frac{1}{2} A(u_0)\,(k)^2$$

$$u^{n+1}k = 2u^n - u^{n-1} + A(u^n)\,(k)^2$$

Using staggered time-grid idea we can write it as a system by introducing sequence of "velocities":

$$v^0 := v_0 + \frac{1}{2} A(x_0) k \text{ and } v^{n+1/2} := \frac{u^{n+1} - u^n}{k}, \text{ for each } n = 1, \ldots, N-1.$$



Leapfrog for wave

(63.1) $$\partial_{tt} u + \mathscr{A} u = f,\ u(0) = u_0 \text{ and } v(0) = v_0,$$

where $\mathscr{A}$ is elliptic operator (including boundary conditions).

$$\partial_{tt}u + \mathscr{A}u = f,\ u(0) = u_0 \text{ and } v(0) = v_0, \tag{64.1}$$

where $\mathscr{A}$ is elliptic operator (including boundary conditions). We can now write the method (semidiscrete in time) as a system in the staggered form considered in Hairer, Lubich and Wanner, 2003:

$$\begin{aligned} \partial U^n - V^{n-1/2} &= 0, \\ \partial V^{n-1/2} + \mathscr{A}U^{n-1} &= f^{n-1}, \end{aligned} \tag{64.2}$$

for $n = 1, \ldots, N$.

Consider $U : [-k_1, T] \to \operatorname{Dom} \mathscr{A}$ continuous piecewise affine interpolant of $(U^n)_{n \in n=-1,\dots,N}$ then $U_1 : [0, T] \to \operatorname{Dom} \mathscr{A}$ the same for the values $U^{n-1/2} := U(t_{n-1/2})$, for $n = 0, \dots, N$. Similarly, but on the staggered mesh build functions V and V_1.

(65.1)
$$\partial_t U - I_0 V_1 = R_V,$$
$$\partial_t V + \mathscr{A} \tilde{I}_0 U_1 = \tilde{I}_0 f + R_U,$$

where we define the interpolators

(65.2)
$\tilde{I}_0$: piecewise constant midpoint interpolator on $\{(t^{n-1/2}, t^{n+1/2}]\}_{n=0}^{N-1}$,

I_0 piecewise constant midpoint interpolator on $\{(t^{n-1}, t^n]\}_{n=1}^{N-1}$.

Time reconstructions

(66.1) $$\hat{V}(t) := V^{n-1/2} + \int_{t_{n-1/2}}^{t} (-\mathscr{A}U_1 + \tilde{I}_1 f + \rho_U),$$

and

(66.2) $$\hat{U}(t) := U^{n-1} + \int_{t_{n-1}}^{t} (V_1 + \rho_V).$$

Setting $\hat{e}_U := u - \hat{U}$ and $\hat{e}_V := u' - \hat{V}$, we deduce

(66.3) $$\begin{aligned} \hat{e}_V' + \mathcal{A}\hat{e}_U &= \mathcal{R}_1 + \mathcal{R}_f \\ \hat{e}_U' - \hat{e}_V &= \mathcal{R}_2, \end{aligned}$$

with the following residuals

(66.4) $$\begin{aligned} \mathcal{R}_1 &:= -\mathcal{A}(\hat{U} - U_1) - \rho_U, \\ \mathcal{R}_2 &:= \hat{V} - V_1 - \rho_V, \\ \mathcal{R}_f &:= f - \tilde{I}_1 f. \end{aligned}$$

Energy estimates

(67.1)

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||(\hat{e}_U,\hat{e}_V)|||^2 &= \left(\left(\left(\hat{e}_U',\hat{e}_V'\right),\left(\hat{e}_U,\hat{e}_V\right)\right)\right)\\
&= \left(\mathcal{A}\hat{e}_U',\hat{e}_U\right)+\left(\hat{e}_V',\hat{e}_V\right)\\
&= (\mathcal{A}\hat{e}_V,\hat{e}_U)+(\mathcal{A}\mathcal{R}_2,\hat{e}_U)-(\mathcal{A}\hat{e}_U,\hat{e}_V)\\
&\quad +(\mathcal{R}_1,\hat{e}_V)+(\mathcal{R}_f,\hat{e}_V)\\
&= (\mathcal{A}\mathcal{R}_2,\hat{e}_U)+(\mathcal{R}_1,\hat{e}_V)+(\mathcal{R}_f,\hat{e}_V)\,,
\end{aligned}
$$

Final result

Theorem

Let u be the solution of the wave equation, $\hat{e}_U := u - \hat{U}$ and $\hat{e}_V := u' - \hat{V}$. Then, the following a posteriori error estimate holds (68.1)

$$\sup_{t\in[0,t^N]} |||(\hat{e}_U, \hat{e}_V)(t)|||^2 \leq 2|||(\hat{e}_U, \hat{e}_V)(0)|||^2 + 4\Big(\int_0^{t^N} |||(\mathcal{R}_2, \mathcal{R}_1 + \mathcal{R}_f)|||\mathrm{d}t\Big)^2,$$

where $\mathcal{R}_2, \mathcal{R}_1$, and $\mathcal{R}_f$ are defined in (66.4).

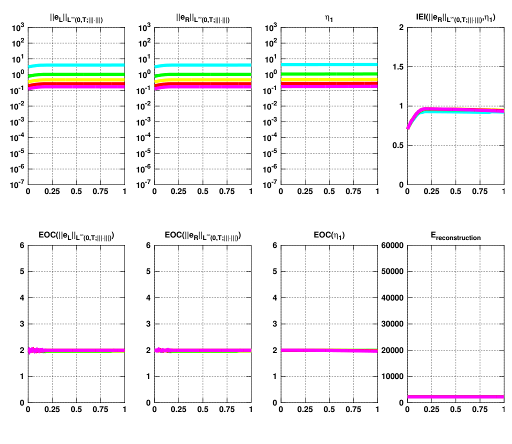
An immediate Corollary is an posteriori bound for the error $\sup_{[0,t^N]} |||(u - U, u' - V)|||$.

Numerics with known exact solution

$u(t,x) := \sum_{j,k=1}^{3} \sin(\pi k x)\sin(\pi j y)\left(\alpha_{k,j}\cos(\pi\xi_{k,j}t) + \beta_{k,j}\sin(\xi_{k,j}\pi t)\right)$

numerics for $c = 1.0, \alpha_{1,1} = \beta_{1,1} = 15.0, \beta_{k,j} = \alpha_{k,j} = 0$

Errors, estimator and inverse effectivity indexes (IEIs) are plotted on the top row while experimental orders of convergence (EOCs) and energy of the reconstructed solution on the bottom row over time (abscissa). Results are computed on the sequence of uniform meshes with mesh size h, fixed time step $k = 0.4h/(p+1)^2$ and $p = 2$. The IEI behaviour indicates that the error is well estimated by the estimator and the convergence rate of the estimator remains near to EOC ≈ 2, i.e., to that of the errors e_L and e_R.

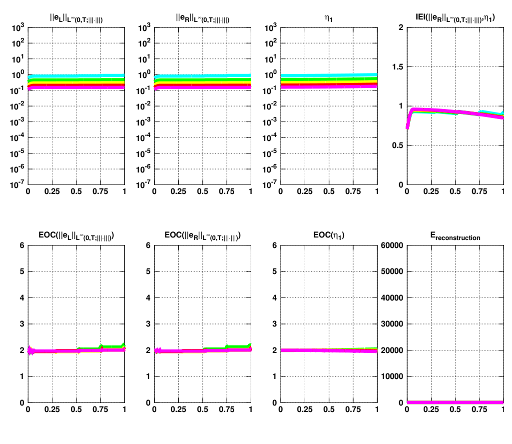


Numerics with known exact solution

$u(t,x) := \sum_{j,k=1}^{3} \sin(\pi k x)\sin(\pi j y)\left(\alpha_{k,j}\cos(\pi\xi_{k,j}t) + \beta_{k,j}\sin(\xi_{k,j}\pi t)\right)$

numerics for $c = 1.0, \alpha_{3,3} = \beta_{3,3} = 1.0, \beta_{k,j} = \alpha_{k,j} = 0$

Errors, estimator and inverse effectivity index (IEI) are plotted on the top row while experimental order of convergence (EOC) and energy of the reconstructed solution on the bottom row against time in abscissa. Results are computed on the sequence of uniform meshes with mesh size h, fixed time step $k = 0.4h/(p+1)^2$ and $p = 2$. The IEI behaviour indicates that the error is well estimated by the estimator and the convergence rate of the estimator remains near 2, i.e., near that of the errors e_L and e_R.

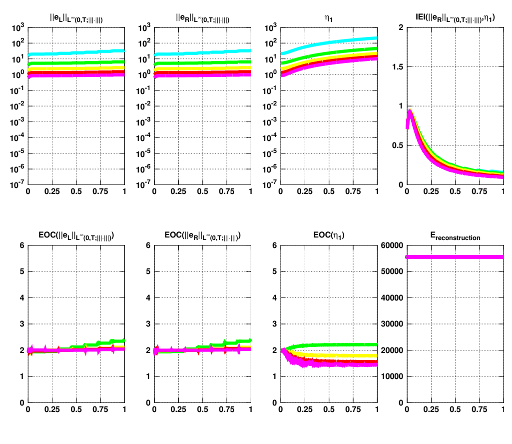


Numerics with known exact solution

$u(t,x) := \sum_{j,k=1}^{3} \sin(\pi k x)\sin(\pi j y)\left(\alpha_{k,j}\cos(\pi\xi_{k,j}t) + \beta_{k,j}\sin(\xi_{k,j}\pi t)\right)$

numerics for $c = 5.0, \alpha_{1,1} = \beta_{1,1} = 15.0, \beta_{k,j} = \alpha_{k,j} = 0$

Errors, estimator and inverse effectivity indexes (IEIs) are depicted on the top row, while experimental orders of convergence (EOCs) and energy of the reconstructed solution on the bottom row over time (x-axis). Results are computed on the sequence of uniform meshes with mesh size h, time-step $k = 0.1h/(p+1)^2$ and $p = 2$. The IEI behaviour indicates that the error is overestimated by the estimator.

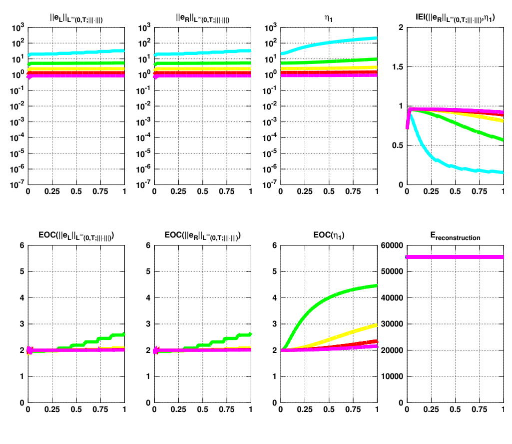


Numerics with known exact solution

$u(t,x) := \sum_{j,k=1}^{3} \sin(\pi k x)\sin(\pi j y)\left(\alpha_{k,j}\cos(\pi\xi_{k,j}t) + \beta_{k,j}\sin(\xi_{k,j}\pi t)\right)$

numerics for $c = 5.0, \alpha_{1,1} = \beta_{1,1} = 15.0, \beta_{k,j} = \alpha_{k,j} = 0$

Errors, estimator and IEI are depicted on the top row and EOCs and energy of the reconstructed solution on the bottom row over time in abscissa. Results are computed on the sequence of uniform meshes with mesh size h, time-step $k = 0.4h^2/(p+1)^2$, and $p = 2$.

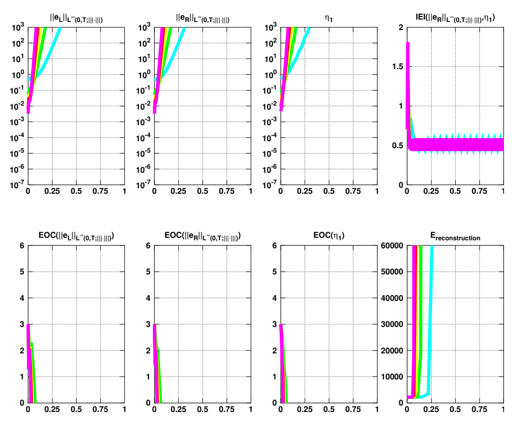


Numerics with known exact solution

$u(t,x) := \sum_{j,k=1}^{3} \sin(\pi k x)\sin(\pi j y)\left(\alpha_{k,j}\cos(\pi\xi_{k,j}t) + \beta_{k,j}\sin(\xi_{k,j}\pi t)\right)$

numerics for $c = 1.0, \alpha_{1,1} = \beta_{1,1} = 15.0, \beta_{k,j} = \alpha_{k,j} = 0$ violation of the CFL condition

Violation of the CFL condition. Errors, estimator and IEI are depicted on the top row and EOCs and energy of the reconstructed solution on the bottom row against time in abscissa. Results are computed on the sequence of uniform meshes with mesh size h and time step $k = 2.0h/(p+1)^2$ and $p = 3$. The IEI's behaviour indicates that the error is overestimated by the estimator but follows the error behaviour.



Conclusions

1. H_0^1(space) based adaptive Galerkin methods for the Wave Equation dealt with by Bernardi and Süli, 2005, W. Bangerth, Geiger and R. Rannacher, 2010, Picasso, 2010 & others.

Conclusions

1. H^1_0(space) based adaptive Galerkin methods for the Wave Equation dealt with by Bernardi and Süli, 2005, W. Bangerth, Geiger and R. Rannacher, 2010, Picasso, 2010 & others.
2. L_2(space) based aposteriori error estimates first derived in wave equation (Georgoulis, Lakkis and Makridakis, 2013).

Conclusions

1. H_0^1(space) based adaptive Galerkin methods for the Wave Equation dealt with by Bernardi and Süli, 2005, W. Bangerth, Geiger and R. Rannacher, 2010, Picasso, 2010 & others.
2. L_2(space) based aposteriori error estimates first derived in wave equation (Georgoulis, Lakkis and Makridakis, 2013).
3. Elliptic reconstruction allows the splitting of space and time in a way to allow a careful study of time discretization.

Conclusions

1. H_0^1(space) based adaptive Galerkin methods for the Wave Equation dealt with by Bernardi and Süli, 2005, W. Bangerth, Geiger and R. Rannacher, 2010, Picasso, 2010 & others.
2. L_2(space) based aposteriori error estimates first derived in wave equation (Georgoulis, Lakkis and Makridakis, 2013).
3. Elliptic reconstruction allows the splitting of space and time in a way to allow a careful study of time discretization.
4. Time reconstruction is not trivial:

Conclusions

1. H^1_0(space) based adaptive Galerkin methods for the Wave Equation dealt with by Bernardi and Süli, 2005, W. Bangerth, Geiger and R. Rannacher, 2010, Picasso, 2010 & others.
2. L_2(space) based aposteriori error estimates first derived in wave equation (Georgoulis, Lakkis and Makridakis, 2013).
3. Elliptic reconstruction allows the splitting of space and time in a way to allow a careful study of time discretization.
4. Time reconstruction is not trivial:
   a. Baker's trick needs tweaking to go through,

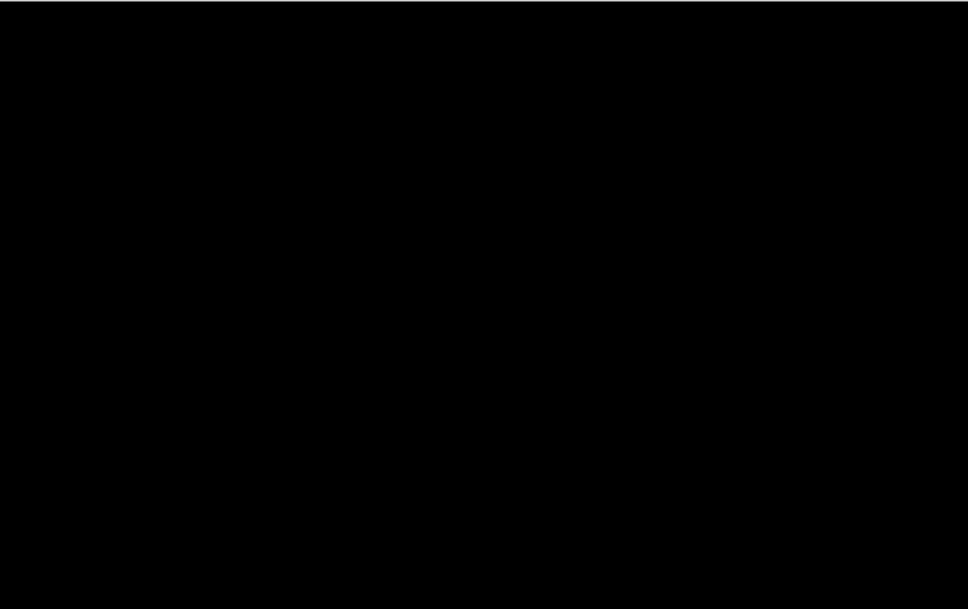
Conclusions

1. H^1_0(space) based adaptive Galerkin methods for the Wave Equation dealt with by Bernardi and Süli, 2005, W. Bangerth, Geiger and R. Rannacher, 2010, Picasso, 2010 & others.
2. L_2(space) based aposteriori error estimates first derived in wave equation (Georgoulis, Lakkis and Makridakis, 2013).
3. Elliptic reconstruction allows the splitting of space and time in a way to allow a careful study of time discretization.
4. Time reconstruction is not trivial:
   - a. Baker's trick needs tweaking to go through,
   - b. "local vanishing moment factor" μ^n has to be introduced in order to get the correctly balanced time-estimator.

Conclusions

1. H^1_0(space) based adaptive Galerkin methods for the Wave Equation dealt with by Bernardi and Süli, 2005, W. Bangerth, Geiger and R. Rannacher, 2010, Picasso, 2010 & others.
2. L_2(space) based aposteriori error estimates first derived in wave equation (Georgoulis, Lakkis and Makridakis, 2013).
3. Elliptic reconstruction allows the splitting of space and time in a way to allow a careful study of time discretization.
4. Time reconstruction is not trivial:
   - a. Baker's trick needs tweaking to go through,
   - b. "local vanishing moment factor" μ^n has to be introduced in order to get the correctly balanced time-estimator.
5. Practical estimates must be derived for practical schemes. Backward Euler is an academic exercise, but leapfrog/Verlet and more generally cosine/Newmark methods must be analyzed. A first step by Georgoulis, Lakkis, Makridakis and Virtanen (2014) with the staggered timestepping.

Credits



Credits

- Charalambos “Babis” Makridakis (Crete & Sussex): main theory, fully discrete

Credits

- Charalambos "Babis" Makridakis (Crete & Sussex): main theory, fully discrete

Credits

- Charalambos "Babis" Makridakis (Crete & Sussex): main theory, fully discrete
- Emmanuil "Manolis" Georgoulis (Leicester & Athens): DG and wave

Credits

- Charalambos “Babis” Makridakis (Crete & Sussex): main theory, fully discrete
- Emmanuil “Manolis” Georgoulis (Leicester & Athens): DG and wave

Credits

- Charalambos "Babis" Makridakis (Crete & Sussex): main theory, fully discrete
- Emmanuil "Manolis" Georgoulis (Leicester & Athens): DG and wave
- Juha Virtanen (Leicester):

Credits

- Charalambos "Babis" Makridakis (Crete & Sussex): main theory, fully discrete
- Emmanuil "Manolis" Georgoulis (Leicester & Athens): DG and wave
- Juha Virtanen (Leicester):
- Taxpayers:

Credits

- Charalambos "Babis" Makridakis (Crete & Sussex): main theory, fully discrete
- Emmanuil "Manolis" Georgoulis (Leicester & Athens): DG and wave
- Juha Virtanen (Leicester):
- Taxpayers:
  - EPSRC (UK)

Credits

- Charalambos "Babis" Makridakis (Crete & Sussex): main theory, fully discrete
- Emmanuil "Manolis" Georgoulis (Leicester & Athens): DG and wave
- Juha Virtanen (Leicester):
- Taxpayers:
  - EPSRC (UK)
  - Hausdorff Institute Bonn (DE)

Baker, Garth A. (1976). "Error estimates for finite element methods for second order hyperbolic equations". In: SIAM J. Numer. Anal. 13.4, pp. 564–576. ISSN: 0036-1429.

Bangerth, W., M. Geiger and R. Rannacher (2010). "Adaptive Galerkin finite element methods for the wave equation". In: Comput. Methods Appl. Math. 10.1, pp. 3–48. ISSN: 1609-4840. DOI: 10.2478/cmam-2010-0001. URL: http://dx.doi.org/10.2478/cmam-2010-0001.

Bangerth, W. and R. Rannacher (1999). "Finite element approximation of the acoustic wave equation: error control and mesh adaptation". In: East-West J. Numer. Math. 7.4, pp. 263–282. ISSN: 0928-0200.

Bangerth, Wolfgang and Rolf Rannacher (2001). "Adaptive finite element techniques for the acoustic wave equation". In: J. Comput. Acoust. 9.2, pp. 575–591. ISSN: 0218-396X. DOI: 10.1142/S0218396X01000668. URL: http://dx.doi.org/10.1142/S0218396X01000668.

Bernardi, Christine and Endre Süli (2005). "Time and space adaptivity for the second-order wave equation". In: Math. Models Methods Appl. Sci. 15.2, pp. 199–225. ISSN: 0218-2025.

Georgoulis, Emmanuil H., Omar Lakkis and Charalambos Makridakis (2013). "A posteriori $L^\infty(L^2)$-error bounds for finite element approximations to the wave equation". In: IMA J. Numer. Anal. 33.4, pp. 1245–1264. ISSN: 0272-4979. DOI: 10.1093/imanum/drs057. URL: http://arxiv.org/abs/1003.3641.

Georgoulis, Emmanuil H., Omar Lakkis, Charalambos Makridakis and Juha M. Virtanen (2014). A posteriori error estimates for leap-frog and cosine methods for second order evolution problems. Tech. rep. arXiv: 1411.7572. URL: http://arxiv.org/abs/1411.7572 (visited on 24/08/2015).

Hairer, Ernst, Christian Lubich and Gerhard Wanner (2003). “Geometric numerical integration illustrated by the Störmer-Verlet method”. In: Acta Numerica 12, pp. 399–450. ISSN: 1474-0508. DOI: 10.1017/S0962492902000144. URL: http://journals.cambridge.org/article_S0962492902000144 (visited on 03/03/2015).

Picasso, Marco (2010). “Numerical Study of an Anisotropic Error Estimator in the $L^2(H^1)$ Norm for the Finite Element Discretization of the Wave Equation”. In: SIAM Journal on Scientific Computing 32.4, pp. 2213–2234. DOI: 10.1137/090778249. URL: http://doi.dx.org/10.1137/090778249.