Maximum-norm a posteriori estimates on anisotropic meshes

Natalia Kopteva

University of Limerick, Ireland

partially supported by DAAD (Study Visit Grant for Senior Academics)

and Science Foundation Ireland

For singularly perturbed *semilinear reaction-diffusion* equations

$$-\varepsilon^2 \triangle u + f(x,u) = 0$$

where $x \in \Omega \subset \mathbb{R}^2$, subject to $u = 0$ on $\partial\Omega$

$$f(x,u) - f(x,v) \geq C_f[u-v] \text{ whenever } u \geq v, \quad \boxed{\varepsilon^2 + C_f \gtrsim 1}$$

we look for *residual-type a posteriori error estimates*

$$\max_{x \in \bar{\Omega}} \big|\text{error}(x)\big| \leq \text{function}\big(\text{mesh, comp.sol-n}\big)$$

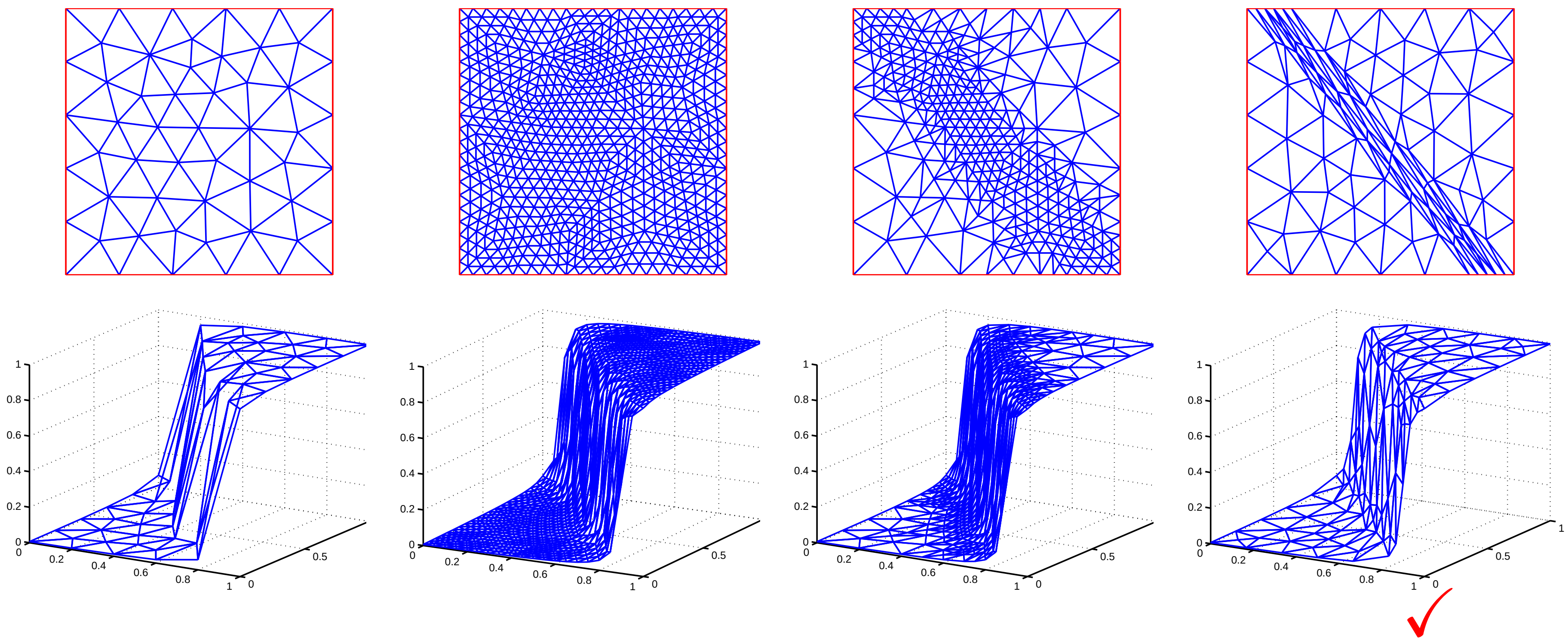
in the *maximum norm*

on **anisotropic meshes**

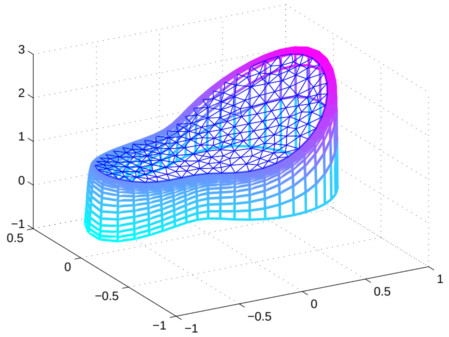
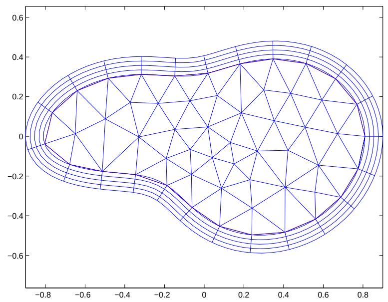
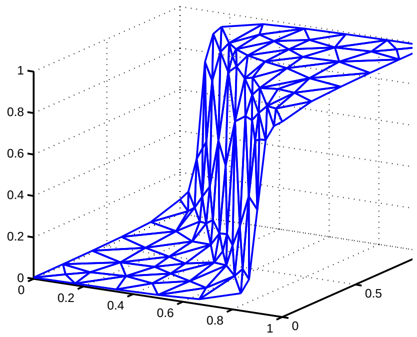
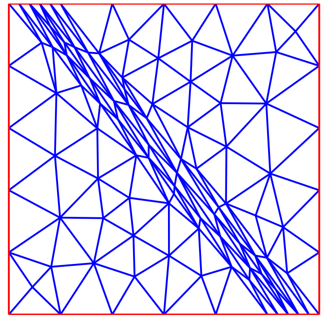
- Interpolation error bounds $\Rightarrow$

anisotropic meshes are superior for layer solutions

(a) Standard mesh. (b) Fine mesh. (c) Shape-regular refinement. (d) *Anisotropic* ref-nt.



- **anisotropic meshes are superior for layer solutions**

(i) **fine in layer regions ; coarse outside**

(ii) **maximum mesh aspect ratio $\sim$ (layer width)$^{-1} \gg 1$**

BUT **theoretical difficulties within the FEM framework...**

Part 0 **Perceptions & expectations t.b. adjusted** for anisotropic meshes

Part 1 **A posteriori estimates on anisotropic meshes**

— Problem addressed (more detail)

— Existing literature

— Mesh assumptions + preview of results

Part 2 **A bit of analysis**: 3 technical issues addressed

1. Application of a Scaled Trace theorem when estimating the Jump Residual ("long" edges cause problems...)
2. Shaper bounds for the Interior Residual (by identifying connected paths of anisotropic nodes...)
3. Quasi-interpolants (of Clément/Scott-Zhang type) are not readily available for general anisotropic meshes [Apel, Chapt. III]...

Part 3 **Numerics. Current+future work** (3d; non-singularly perturbed case...)

One Perception: **the computed-solution error in the maximum norm is closely related to the corresponding interpolation error...**

- Quasi-uniform meshes, linear elements

$$\|u - u_h\|_{L_\infty(\Omega)} \leq \ln(C + \varepsilon/h) \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty(\Omega)}$$

  - Schatz, Wahlbin, *On the quasi-optimality in L_∞ of the $\mathring{H}^1$-projection into finite element spaces*, Math. Comp. 1982: $-\triangle u = f$,
  - Schatz, Wahlbin, *On the finite element method for singularly perturbed reaction-diffusion problems* ..., Math. Comp., 1983: $-\varepsilon^2 \triangle u + au = f$,

One Perception: **the computed-solution error in the maximum norm is closely related to the corresponding interpolation error...**

- Quasi-uniform meshes, linear elements

$$\|u-u_h\|_{L_\infty(\Omega)} \le \ln(C+\varepsilon/h) \inf_{\chi\in S_h} \|u-\chi\|_{L_\infty(\Omega)}$$

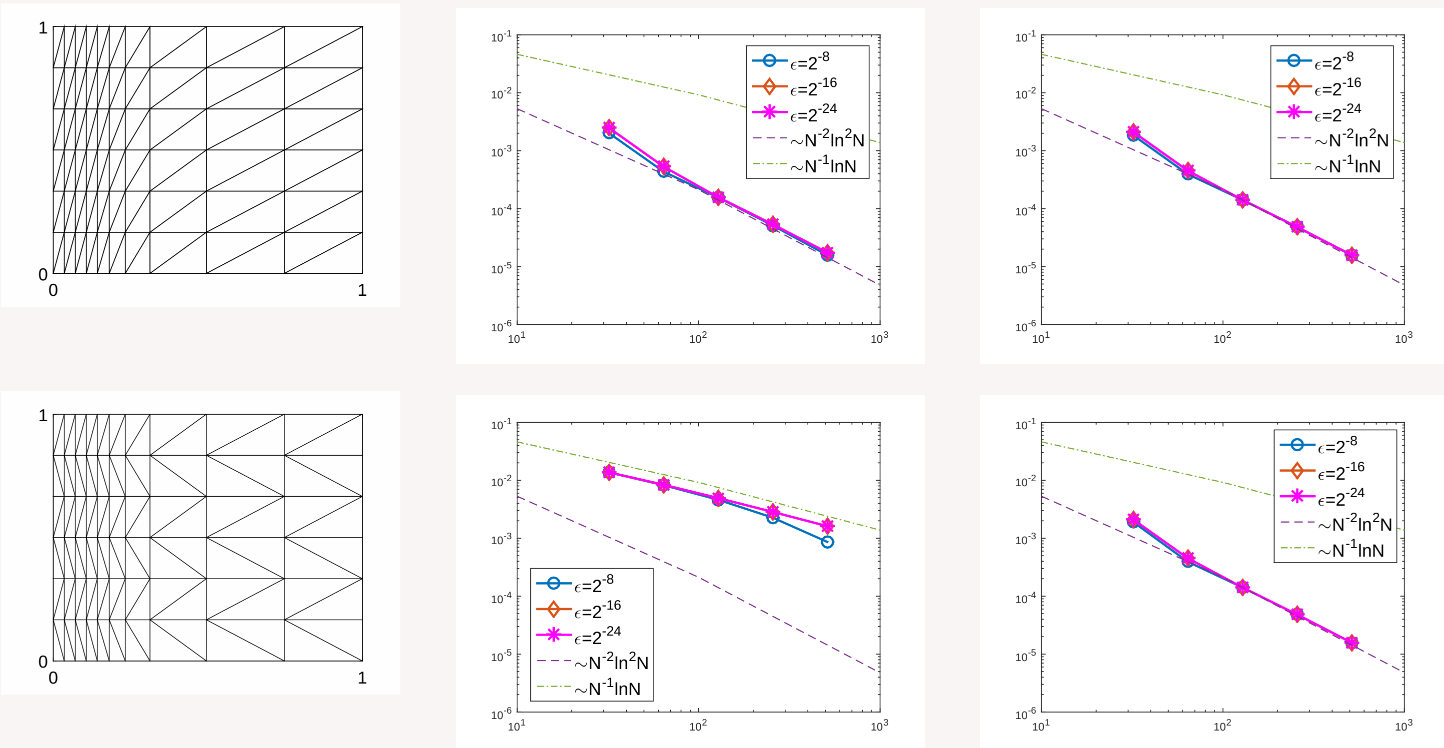
  - Schatz, Wahlbin, *On the quasi-optimality in L_∞ of the $\mathring{H}^1$-projection into finite element spaces*, Math. Comp. 1982: $-\triangle u = f$,
  - Schatz, Wahlbin, *On the finite element method for singularly perturbed reaction-diffusion problems* ..., Math. Comp., 1983: $-\varepsilon^2\triangle u + au = f$,

- **Strongly-anisotropic triangulations:** no such result
  - BUT this is frequently considered a reasonable heuristic conjecture t.b. used in the anisotropic mesh adaptation (Hessian-related metrics...)
  - IN FACT, this is **NOT true** (see next)

<u>Example:</u> $-\varepsilon^2\Delta u + u = 0$ with $u = e^{-x/\varepsilon}$ exhibiting a sharp boundary layer

<u>Observation #1:</u> **Mass Lumping may be superior on anisotropic meshes**

Standard linear FEM | Mass Lumping



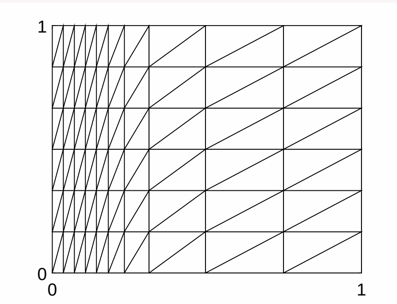
Here we use a Shishkin mesh: piecewise-uniform, $DOF \simeq N^2$, mesh diameter $\simeq N^{-1}$

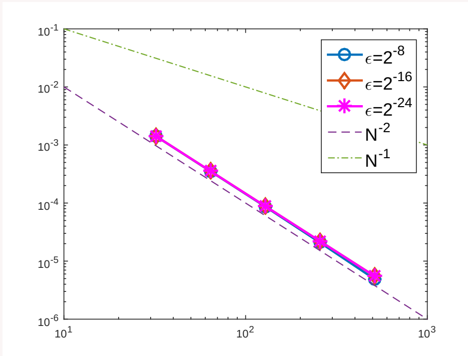
$$\|u - u^I\|_{L_\infty(\Omega)} \simeq N^{-2}\ln^2 N \simeq DOF^{-1}\ln(DOF)$$

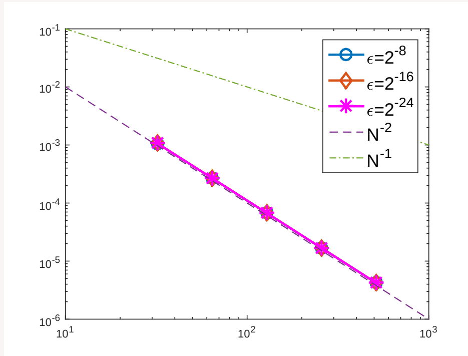
Same Example: $-\varepsilon^2 \triangle u + u = 0$ with $u = e^{-x/\varepsilon}$ exhibiting a **sharp boundary layer**

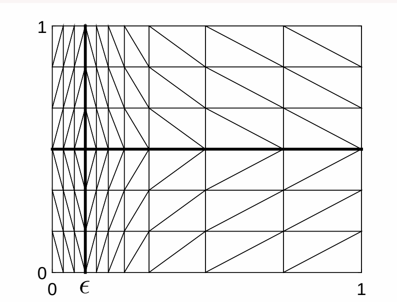
Observation #2: **Convergence Rates may depend on the mesh structure** (even for mass lumping), **NOT ONLY on the interpolation error**

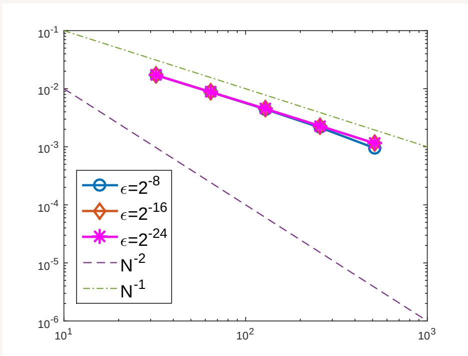
Standard linear FEM | Mass Lumping

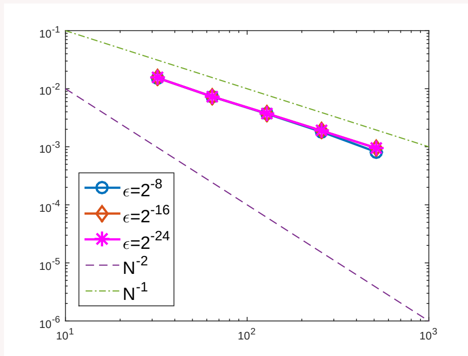












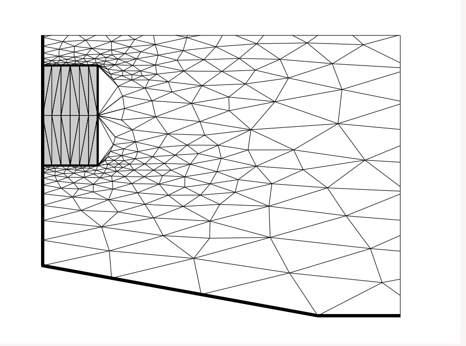
Here we use a graded Bakhvalov mesh:

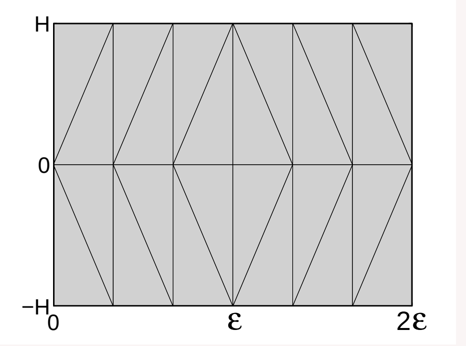
$$\|u - u^I\|_{L_\infty(\Omega)} \simeq N^{-2} \simeq DOF^{-1}$$

- A theoretical explanation of the above phenomena is given in:

  N.Kopteva, *Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations*, Math. Comp., 2014.

What happens in $\mathring{\Omega} := (0, 2\varepsilon) \times (-H, H)$
with the tensor-product mesh $\mathring{\omega}_h := \{x_i = \varepsilon \frac{i}{N_0}\}_{i=0}^{2N_0} \times \{-H, 0, H\}$??

$\mathcal{T}$ in Ω: 

$\mathcal{T}_0$ in $\Omega_0 \subset \Omega$: 

Mass lumping, $U_i := u_h(x_i, 0)$ and $U_i^{\pm} := u_h(x_i, \pm H)$:

$$\frac{\varepsilon^2}{h^2}[-U_{i-1} + 2U_i - U_{i+1}] + \frac{\varepsilon^2}{H^2}[-U_i^- + 2U_i - U_i^+] + \gamma_{\mathbf{i}} U_i = 0$$

with $\gamma_{\mathbf{i}} = \mathbf{1}$ for $i \neq N_0$, and $\gamma_{\mathbf{N_0}} = \mathbf{\frac{2}{3}}$

$$\varepsilon \ll \mathbf{H} \quad \Rightarrow \quad \frac{\varepsilon^2}{h^2}[-U_{i-1} + 2U_i - U_{i+1}] + \frac{\varepsilon^2}{H^2}[-U_i^- + 2U_i - U_i^+] + \gamma_{\mathbf{i}} U_i = 0$$

Implications of the above example:

- Theoretical:

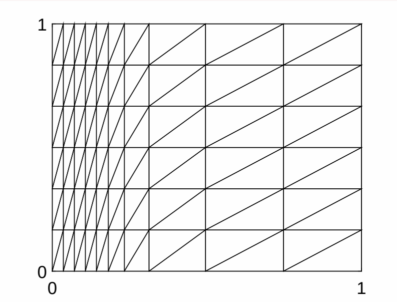
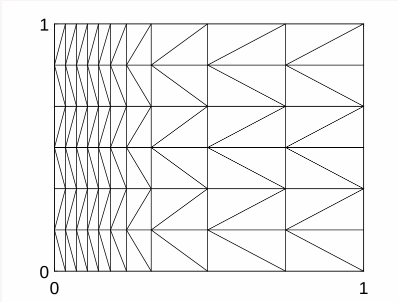
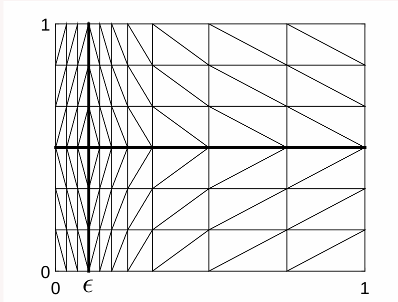
  if one tries to prove ”standard” (almost) second-order a priori/a posteriori error estimate in the maximum norm on a general anisotropic mesh, this may be impossible...

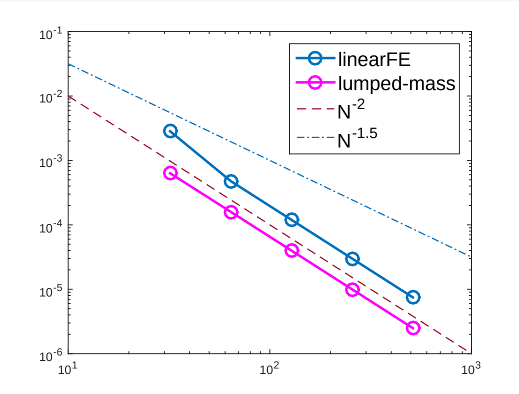
- Anisotropic mesh adaptation (Hessian-related metrics...):

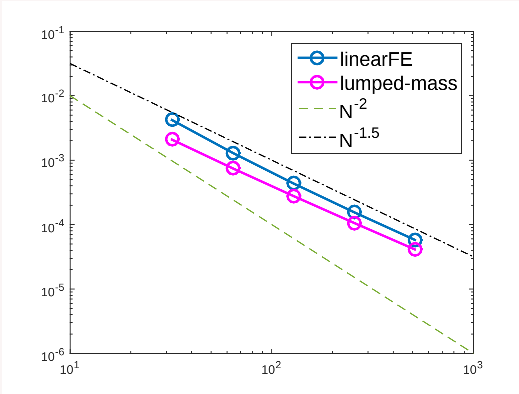
  One needs to be careful with the heuristic conjecture that the computed-solution error in the maximum norm is closely related to the corresponding interpolation error...

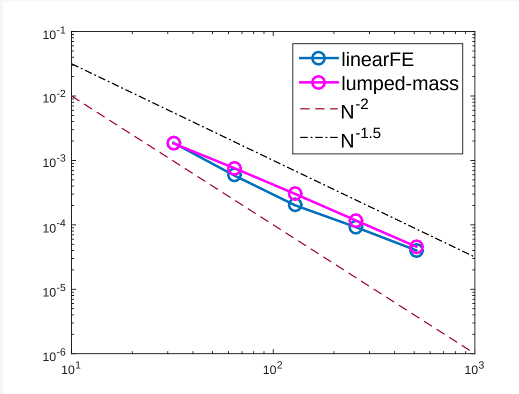
Non-singularly-perturbed EXAMPLE [Nochetto et al, Numer. Math., 2006]:

$$-\triangle u + f(u) = 0 \text{ with } f(u) \sim -u^{-3} \text{ and } u = \sqrt{x}$$







Graded mesh: $\{(i/N)^6\}_{i=0}^{N}$:

$$\|u - u^I\|_{L_\infty(\Omega)} \simeq N^{-2} \simeq DOF^{-1}$$

Mesh transition parameter: $\epsilon = 0.1$

Part 0 **Perceptions & expectations t.b. adjusted** for anisotropic meshes

Part 1 **A posteriori estimates on anisotropic meshes**

— Problem addressed (more detail)

— Existing literature

— Mesh assumptions + preview of results

Part 2 **A bit of analysis**: 3 technical issues addressed

1. Application of a Scaled Trace theorem when estimating the Jump Residual ("long" edges cause problems...)
2. Shaper bounds for the Interior Residual (by identifying connected paths of anisotropic nodes...)
3. Quasi-interpolants (of Clément/Scott-Zhang type) are not readily available for general anisotropic meshes [Apel, Chapt. III]...

Part 3 **Numerics. Current+future work** (3d; non-singularly perturbed case...)

For $-\varepsilon^2 \triangle u + f(x,u) = 0$, we consider a *standard finite element approximation*

$$\varepsilon^2(\nabla u_h, \nabla v_h) + (f_h^I, v_h) = 0, \quad v_h \in S_h, \quad f_h := f(\cdot, u_h)\,,$$

where $S_h \subset H_0^1(\Omega)$ is a linear finite element space

- Ω is a **polygonal, possibly non-Lipschitz,** domain in $\mathbb{R}^n$, $n = 2$:

  $\Rightarrow \quad u \in H_0^1(\Omega) \cap C(\bar{\Omega})$;

  to be more precise, $u \in W_l^2(\Omega) \subseteq W_q^1 \subset C(\bar{\Omega})$ for some $l > \frac{1}{2}n$ and $q > n$.

- one-sided-Lipschitz-condition version of $f_u(x,u) \geq C_f \geq 0$,

  but $f_u \leq \bar{C}_f$ NOT assumed

- **Laplace equation** $-\triangle u = f(x)$

— K. Eriksson, *An adaptive finite element method with efficient maximum norm error control for elliptic problems*, Math. Models Methods Appl. Sci., 4 (1994).

— R. H. Nochetto, *Pointwise a posteriori error estimates for elliptic problems on highly graded meshes*, Math. Comp., 64 (1995).

— E. Dari, R. G. Durán & C. Padra, *Maximum norm error estimators for three-dimensional elliptic problems*, SIAM J. Numer. Anal., 37 (2000).

— A. Demlow & E. Georgoulis, *Pointwise a posteriori error control for discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 50 (2012).

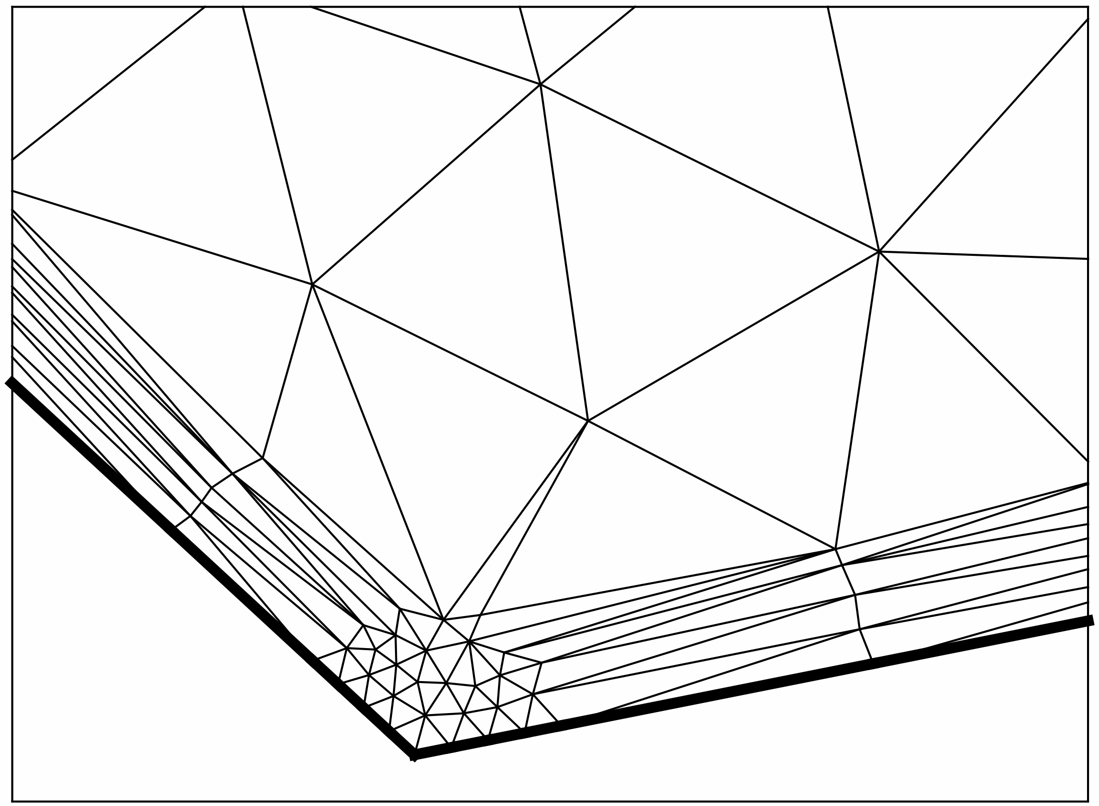
- **Semilinear equation** $-\triangle u + f(x, u) = 0$

— R. H. Nochetto, A. Schmidt, K. G. Siebert, & A. Veeser, *Pointwise a posteriori error estimates for monotone semilinear problems*, Numer. Math., 104 (2006).

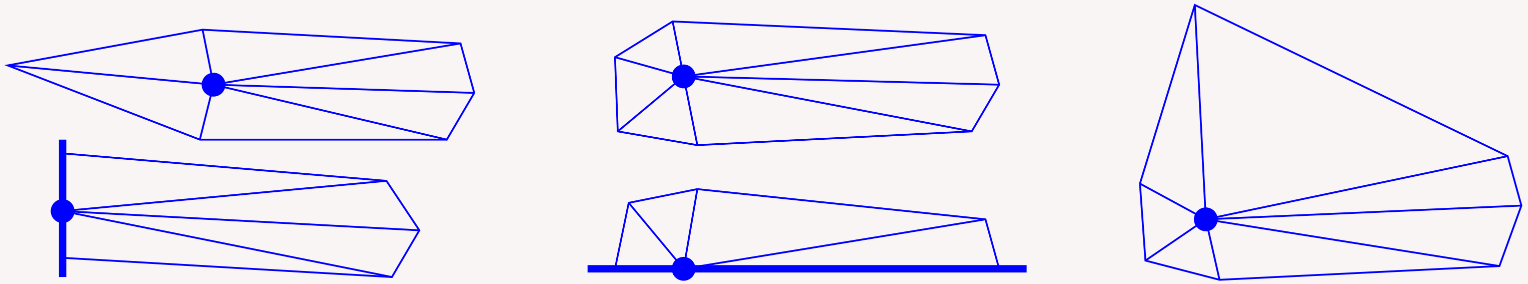
- **Singularly perturbed equation** $-\varepsilon^2 \triangle u + f(x, u) = 0$

— A. Demlow & N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed elliptic reaction-diffusion problems*, Numer. Math., (2015).

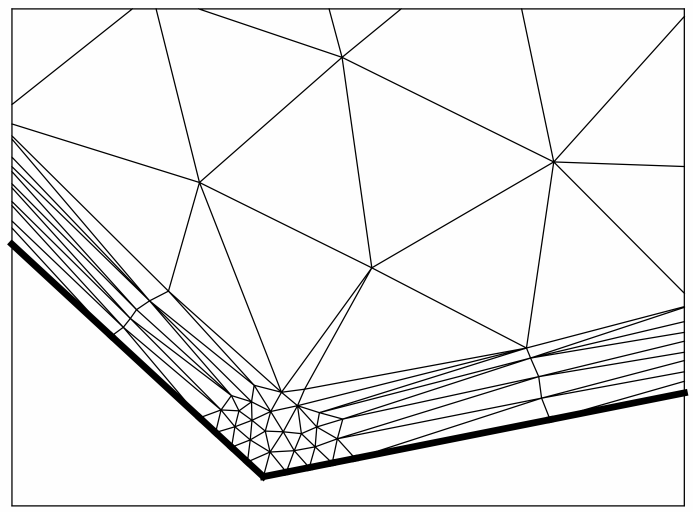
Roughly speaking, want to include meshes of the type:



- Permitted mesh node types:



- Example of a mesh for which the analysis works:



Notation: $H_z := \mathrm{diam}(\omega_z)$, $h_z := \max_{T\subset\omega_z} h_T$, $h_T := 2H_T^{-1}|T|$

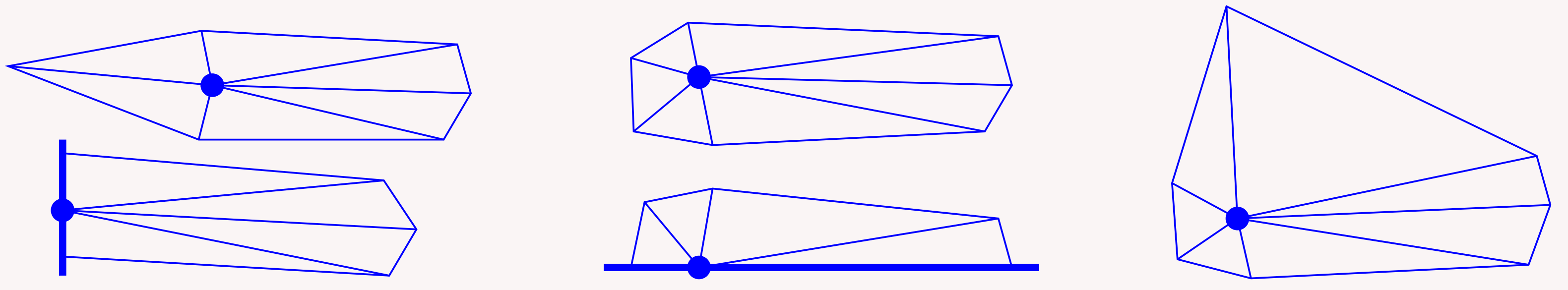
Main Triangulation Assumptions:

- *Maximum Angle condition.*
- *Local Coordinate condition.* For any $z \in \mathcal{N}$, let

$$|\sin\angle(S,\hat{S}_z)| \lesssim \frac{h_z}{|S|} \quad \forall\, S\subset\mathcal{S}_z, \quad \text{where} \quad \hat{S}_z\in\mathcal{S}_z, \quad |\hat{S}_z| = \max_{S\subset\mathcal{S}_z}|S| \qquad (1)$$

- Also let the number of triangles containing any node be uniformly bounded.
- *Quasi-non-obtuse anisotropic elements.* Let the maximum angle in any triangle be bounded by $\frac{\pi}{2} + \alpha_1 \frac{h_T}{H_T}$ for some positive constant α_1.

Mesh Node Types:



Assuming that anisotropic mesh elements are almost non-obtuse,
our FIRST ESTIMATOR reduces to

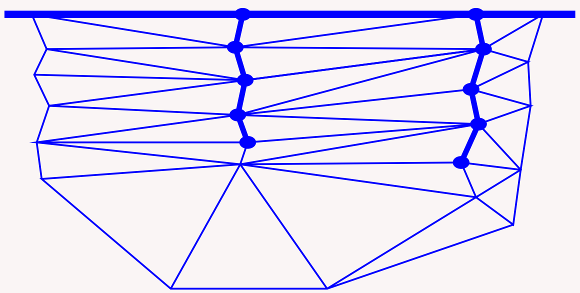
$$\|u_h - u\|_\infty \le C\,\ell_h \max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\,\big\|[\![\nabla u_h]\!]\big\|_{\infty\,;\gamma_z} + \min\big\{1, \tfrac{H_z^2}{\varepsilon^2}\big\}\,\|f_h^I\|_{\infty\,;\omega_z}\Big) + C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,,$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε.

Here $f_h = f(\cdot, u_h)$, $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, $[\![\nabla u_h]\!]$ is the standard jump in the normal derivative of u_h across an element edge, ω_z is the patch of elements surrounding any $z \in \mathcal{N}$, γ_z is the set of edges in the interior of ω_z, $H_z = \mathrm{diam}(\omega_z)$, $\ell_h = \ln(2 + \varepsilon \underline{h}^{-1})$, and $\underline{h}$ is the minimum height of triangles in $\mathcal{T}$.

- For $\varepsilon = 1$, this gives a standard a posteriori error bound, similar to [Eriksson, Nochetto, Nochetto et al], only now we prove it for anisotropic meshes.
- For $\varepsilon \in (0, 1]$, this is almost identical with our estimator for shape-regular case (on the previous page), but now we assume no shape regularity of the mesh.

In order to give a sharper (and more anisotropic in nature) bound for the interior-residual component of the error, we identify sequences of short edges that connect anisotropic nodes:



Under some additional assumptions on each such sequence (which we call a Path), our SECOND ESTIMATOR

$$\|u_h - u\|_\infty \le C\,\ell_h \Big[\max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\,\|J_z\|_{\infty\,;\gamma_z}\Big) + \max_{z\in\mathcal{N}\setminus\mathcal{N}_{\rm paths}}\Big(\min\{1,\varepsilon^{-2}H_z^2\}\,\|f_h^I\|_{\infty\,;\omega_z}\Big)$$
$$+ \max_{z\in\mathcal{N}_{\rm paths}}\Big(\min\{\varepsilon, H_z\}\,\min\{\varepsilon, h_z\}\|\varepsilon^{-2}f_h^I\|_{\infty\,;\omega_z} + \min\{1,\varepsilon^{-2}H_z^2\}\,\mathrm{osc}(f_h^I;\omega_z)\Big)\Big]$$
$$+ C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,,$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε.

Here $\mathcal{N}_{\rm paths}$ is the set of mesh nodes that appear in any path, $h_z \sim H_z^{-1}|\omega_z|$, $J_z = [\![\nabla u_h]\!]$.

Part 0 **Perceptions & expectations t.b. adjusted** for anisotropic meshes

Part 1 **A posteriori estimates on anisotropic meshes**

— Problem addressed (more detail)

— Existing literature

— Mesh assumptions + preview of results

Part 2 **A bit of analysis**: 3 technical issues addressed

1. Application of a Scaled Trace theorem when estimating the Jump Residual ("long" edges cause problems...)
2. Shaper bounds for the Interior Residual (by identifying connected paths of anisotropic nodes...)
3. Quasi-interpolants (of Clément/Scott-Zhang type) are not readily available for general anisotropic meshes [Apel, Chapt. III]...

Part 3 **Numerics. Current+future work** (3d; non-singularly perturbed case...)

- For a solution u and <u>any</u> $u_h \in H_0^1(\Omega) \cap W_1^q(\Omega)$ with $q > n$,

$$\boxed{u_h - u = \varepsilon^2(\nabla u_h, \nabla G) + (f_h, G)}$$

- <u>THEOREM</u> [Demlow, Kopteva, 2015] For any $x \in \Omega$,

$$\|G(x,\cdot)\|_{1;\Omega} + \varepsilon\|\nabla G(x,\cdot)\|_{1;\Omega} \lesssim 1.$$

For the ball $B(x,\varrho)$ of radius ϱ centered at $x \in \Omega$, and $\ell_\varrho := \ln(2 + \varepsilon\varrho^{-1})$,

$$\begin{aligned}
\|G(x,\cdot)\|_{1,B(x,\varrho)\cap\Omega} &\lesssim \varepsilon^{-2}\varrho^2\,\ell_\varrho,\\
\|\nabla G(x,\cdot)\|_{1,B(x,\varrho)\cap\Omega} &\lesssim \varepsilon^{-2}\varrho,\\
\|D^2 G(x,\cdot)\|_{1,\Omega\setminus B(x,\varrho)} &\lesssim \varepsilon^{-2}\ell_\varrho
\end{aligned}$$

NEXT: $$u_h - u = \varepsilon^2(\nabla u_h, \nabla(G - G_h)) + (f_h, G - G_h) \quad \forall G_h \in S_h$$

NEXT: $$u_h - u = \varepsilon^2(\nabla u_h, \nabla(G - G_h)) + (f_h, G - G_h) \quad \forall G_h \in S_h$$

NOTE: by the **Divergence Theorem** for each $T \subset \mathcal{T}$,

$$\int_T \nabla u_h \cdot \nabla(G - G_h)) = \int_{\partial T} (G - G_h)) \, \nabla u_h \cdot \nu - \int_T \triangle u_h \, (G - G_h))$$

SO

$$u_h - u = \sum_{S \in \mathcal{S}} \varepsilon^2 \int_S (G - G_h) [\![\nabla u_h]\!] \cdot \nu + \sum_{T \in \mathcal{T}} \int_T (f_h - \varepsilon^2 \triangle u_h) \, (G - G_h)$$

NEXT: $$u_h - u = \varepsilon^2(\nabla u_h, \nabla(G - G_h)) + (f_h, G - G_h) \quad \forall G_h \in S_h$$

NOTE: by the **Divergence Theorem** for each $T \subset \mathcal{T}$,

$$\int_T \nabla u_h \cdot \nabla(G - G_h)) = \int_{\partial T} (G - G_h))\, \nabla u_h \cdot \nu - \int_T \triangle u_h\, (G - G_h))$$

SO

$$u_h - u = \sum_{S \in \mathcal{S}} \varepsilon^2 \int_S (G - G_h)[\![\nabla u_h]\!] \cdot \nu + \sum_{T \in \mathcal{T}} \int_T (f_h - \varepsilon^2 \triangle u_h)\, (G - G_h)$$

As $\forall G_h \in S_h$, so replace $(G - G_h)$ by

$$G - G_h - \sum_{z \in \mathcal{N}} \bar{g}_z \phi_z = \sum_{z \in \mathcal{N}} [G - G_h - \bar{g}_z]\phi_z$$

where $\phi_z =$ the standard hat function associated with a node z

$$u_h - u = \sum_{z \in \mathcal{N}} \varepsilon^2 \int_{\gamma_z} [G - G_h - \bar{g}_z]\phi_z [\![\nabla u_h]\!] \cdot \nu + \sum_{z \in \mathcal{N}} \int_{\omega_z} f_h\, [G - G_h - \bar{g}_z]\phi_z$$

JUMP RESIDUAL:

$$I := \sum_{z\in\mathcal{N}} \varepsilon^2 \int_{\gamma_z} [G - G_h - \bar{g}_z]\phi_z [\![\nabla u_h]\!] \cdot \nu$$

NOTE: An inspection of standard proofs for shape-regular meshes reveals that one obstacle in extending them to anisotropic meshes lies in the application of a **Scaled Trace Theorem** when estimating the jump residual terms (this causes the mesh aspect ratios to appear in the estimator; **"long" edges** cause this problem).

Scaled Trace Theorem (for anisotropic elements; sharp):

$$\max_{S\in\{\text{short edges}\}} \|v\|_{1\,;S} + \frac{\mathbf{h_z}}{\mathbf{H_z}} \max_{S\in\{\text{long edges}\}} \|v\|_{1\,;S} \lesssim H_z^{-1}\|v\|_{1\,;\omega_z} + \|\nabla v\|_{1\,;\omega_z}$$

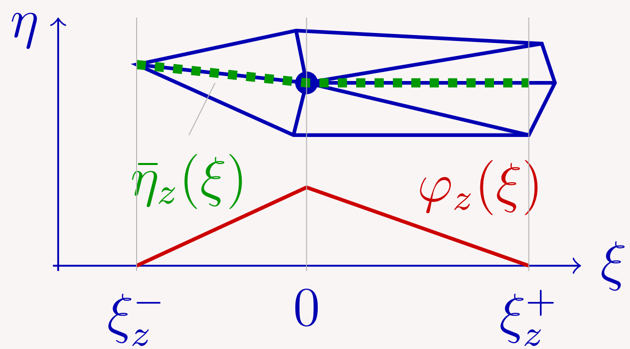
JUMP RESIDUAL:
$$I := \sum_{z\in\mathcal{N}} \varepsilon^2 \int_{\gamma_z} [G - G_h - \bar{g}_z]\phi_z [\![\nabla u_h]\!] \cdot \nu$$

NOTE: An inspection of standard proofs for shape-regular meshes reveals that one obstacle in extending them to anisotropic meshes lies in the application of a **Scaled Trace Theorem** when estimating the jump residual terms (this causes the mesh aspect ratios to appear in the estimator; **”long” edges** cause this problem).

NOTE standard choices: $\bar{g}_z = 0$, or $\int_{\omega_z} (G - G_h - \bar{g}_z)\,\phi_z = 0$ [Nochetto].

Our CHOICE is crucial in addressing this difficulty:

$$\int_{\xi_z^-}^{\xi_z^+} \big[(G - G_h)(\xi, \bar{\eta}_z(\xi)) - \bar{g}_z\big]\, \varphi_z(\xi)\, d\xi = 0$$



Assuming that anisotropic mesh elements are almost non-obtuse ...,
our FIRST ESTIMATOR reduces to

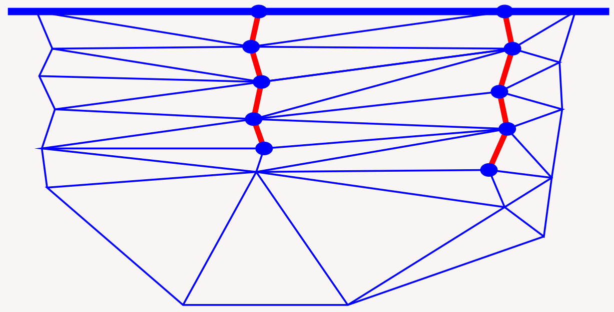
$$\|u_h - u\|_\infty \le C\,\ell_h \max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\,\big\|[\![\nabla u_h]\!]\big\|_{\infty\,;\gamma_z} + \min\{\varepsilon^2, H_z^2\}\,\|\varepsilon^{-2} f_h^I\|_{\infty\,;\omega_z}\Big) + C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,,$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε.

Here $f_h = f(\cdot, u_h)$, $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, $[\![\nabla u_h]\!]$ is the standard jump in the normal derivative of u_h across an element edge, ω_z is the patch of elements surrounding any $z \in \mathcal{N}$, γ_z is the set of edges in the interior of ω_z, $H_z = \mathrm{diam}(\omega_z)$, $\ell_h = \ln(2 + \varepsilon\underline{h}^{-1})$, and $\underline{h}$ is the minimum height of triangles in $\mathcal{T}$.

- For $\varepsilon = 1$, this gives a standard a posteriori error bound, similar to [Eriksson, Nochetto, Nochetto et al], only now we prove it for anisotropic meshes.
- For $\varepsilon \in (0, 1]$, this is almost identical with our estimator for shape-regular case [Demlow, Kopteva], but now we assume no shape regularity of the mesh.

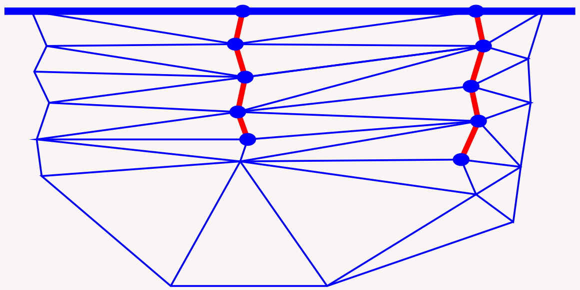
In order to give a sharper (and more anisotropic in nature) bound for the interior-residual component of the error, we identify sequences of short edges that connect anisotropic nodes (and call each of them a Path):



Main Additional Assumption:

- *Path Coordinate-System condition.* For each (semi-)anisotropic path $\mathcal{N}_i$, $i = 1, \ldots, n_{\rm ani} + n_{\rm s.ani}$, let there exist a cartesian coordinate system $(\xi, \eta) = (\xi_i, \eta_i)$ such that $|\sin(\angle(S, {\rm i}_\xi))| \lesssim \frac{h_z}{|S|}$ for any $S \subset \mathcal{S}_z$ of any node $z \in \mathcal{N}_i$ (while, if $\mathcal{N}_i$ is semi-anisotropic a stronger condition $|\angle(S, {\rm i}_\xi)| \lesssim \frac{h_z}{|S|}$ is satisfied).

Let $\mathcal{N}_{\text{paths}}$ be the set of mesh nodes that appear in any path, $h_z \sim H_z^{-1}|\omega_z|$, $J_z = [\![\nabla u_h]\!]$.



SECOND ESTIMATOR

$$\|u_h - u\|_\infty \le C\,\ell_h \Big[\max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\|J_z\|_{\infty\,;\gamma_z}\Big) + \max_{z\in\mathcal{N}\setminus\mathcal{N}_{\text{paths}}}\Big(\min\{1,\varepsilon^{-2}H_z^2\}\|f_h^I\|_{\infty\,;\omega_z}\Big)$$

$$+ \max_{z\in\mathcal{N}_{\text{paths}}}\Big(\min\{\varepsilon, H_z\}\;\min\{\varepsilon, h_z\}\|\varepsilon^{-2}f_h^I\|_{\infty\,;\omega_z} + \min\{1,\varepsilon^{-2}H_z^2\}\,\mathrm{osc}(f_h^I;\omega_z)\Big)\Big]$$

$$+ C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,,$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε.

TASK: estimate

$$\bar{\Theta} := \varepsilon^2 \sum_{T\in\mathcal{T}} \Big(\lambda_T^{-1} \|\nabla(G - G_h)\|_{1\,;T} + \lambda_T^{-2} \|G - G_h\|_{1\,;T} \Big), \ \lambda_T := \min\{\varepsilon, H_T\},$$

Aim: $\bar{\Theta} \lesssim \ell_h$

- It would be convenient to employ a quasi-interpolant (of Clément/Scott-Zhang type) with the property

$$|G - G_h|_{k,p\,;T} \lesssim H_T^{j-k} |G|_{j,p\,;\omega_T} \text{ for any } 0 \le \boxed{k \le j} \le 2, \ p = 1.$$

T.b. more precise, the estimator involves $\min\{\underbrace{1}_{\text{from } k=j}, \underbrace{\frac{H_T^2}{\varepsilon^2}}_{\text{from } k<j}\}$

- However, such interpolants are not readily available for general anisotropic meshes (see [Apel, Chapt. III] for a discussion of Scott-Zhang-type interpolation on anisotropic tensor-product meshes).

TASK: estimate

$$\bar{\Theta} := \varepsilon^2 \sum_{T\in\mathcal{T}} \Big(\lambda_T^{-1}\|\nabla(G-G_h)\|_{1\,;T} + \lambda_T^{-2}\|G-G_h\|_{1\,;T}\Big),\ \ \lambda_T := \min\{\varepsilon, H_T\},$$

- It would be convenient to employ a quasi-interpolant (of Clément/Scott-Zhang type) with the property

$$|G-G_h|_{k,p\,;T} \lesssim H_T^{j-k}|G|_{j,p\,;\omega_T} \text{ for any } 0 \le \boxed{k \le j} \le 2,\ \ p=1.$$

- However, such interpolants are not readily available for general anisotropic meshes (see [Apel, Chapt. III] for a discussion of Scott-Zhang-type interpolation on anisotropic tensor-product meshes).

- Because of this difficulty, we employ a less standard interpolant G_h, which gives a version of the Lagrange interpolant whenever $H_T \lesssim \varepsilon$, and vanishes whenever $H_T \gtrsim \varepsilon$; however, this construction requires additional mild assumptions on the triangulation...

Lemma: $\bar{\Theta} \lesssim \ell_h$

Part 0 **Perceptions & expectations t.b. adjusted** for anisotropic meshes

Part 1 **A posteriori estimates on anisotropic meshes**

— Problem addressed (more detail)

— Existing literature

— Mesh assumptions + preview of results

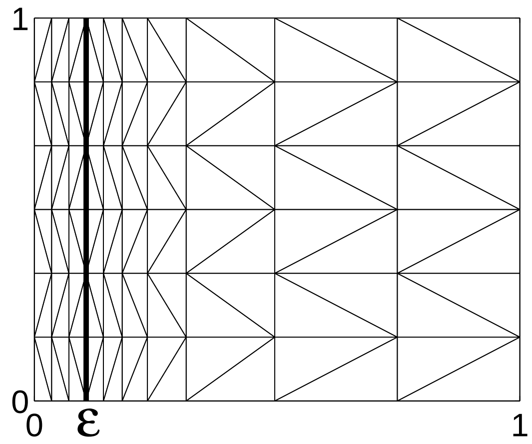
Part 2 **A bit of analysis**: 3 technical issues addressed

1. Application of a Scaled Trace theorem when estimating the Jump Residual ("long" edges cause problems...)
2. Shaper bounds for the Interior Residual (by identifying connected paths of anisotropic nodes...)
3. Quasi-interpolants (of Clément/Scott-Zhang type) are not readily available for general anisotropic meshes [Apel, Chapt. III]...

Part 3 **Numerics. Current+future work** (3d; non-singularly perturbed case...)

Simple 2d TEST problem: $-\varepsilon^2 \Delta u + u = F(x)$ in $\Omega = (0,1)^2$ with $\varepsilon^2 = 10^{-6}$,
$u = 4y\,(1-y)\,[1 - x^2 - (e^{-x/\varepsilon} - e^{-1/\varepsilon})/(1 - e^{-x/\varepsilon})]$

We consider one a-priori-chosen layer-adapted mesh of Bakhvalov type:



- The mesh is chosen so that the linear interpolation error $\|u - u^I\|_{\infty\,;\Omega} \lesssim N^{-2}$.
- However, **as $\varepsilon \to 0$, the convergence rates deteriorate from 2 to 1.**

  This phenomenon is noted and explained in

  [N. Kopteva, *Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations*, Math. Comp. 2014.].

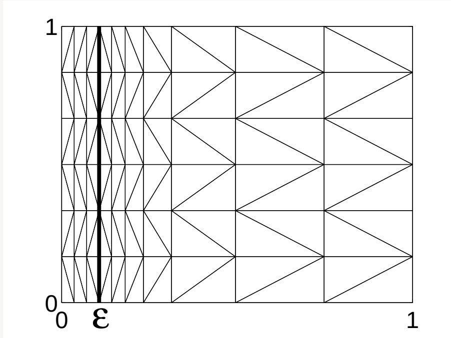
Table: Bakhvalov mesh, $M = \frac{1}{2}N$: maximum nodal errors and estimators.

N	$\varepsilon = 1$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-10}$	$\varepsilon = 2^{-15}$	$\varepsilon = 2^{-20}$	$\varepsilon = 2^{-25}$	$\varepsilon = 2^{-30}$
	Errors (odd rows) & Computational Rates (even rows)						
64	3.373e-4	3.723e-3	8.952e-3	8.973e-3	8.973e-3	8.973e-3	8.973e-3
	2.00	1.91	1.01	1.00	1.00	1.00	1.00
128	8.445e-5	9.935e-4	4.446e-3	4.484e-3	4.484e-3	4.484e-3	4.484e-3
	2.00	1.98	1.04	1.00	1.00	1.00	1.00
256	2.112e-5	2.523e-4	2.165e-3	2.236e-3	2.236e-3	2.236e-3	2.236e-3
	FIRST Estimator (odd rows) & Effectivity Indices (even rows)						
64	6.810e-3	2.516e-1	9.403e-1	9.981e-1	9.999e-1	1.000e+0	1.000e+0
	20.19	67.59	105.04	111.23	111.44	111.45	111.45
128	1.761e-3	1.120e-1	8.858e-1	9.961e-1	9.999e-1	1.000e+0	1.000e+0
	20.86	112.72	199.26	222.15	222.98	223.01	223.01
256	4.480e-4	4.036e-2	7.901e-1	9.922e-1	9.998e-1	1.000e+0	1.000e+0
	21.21	159.97	365.01	443.82	447.17	447.27	447.28

Table: Bakhvalov mesh, $M = \frac{1}{2}N$: maximum nodal errors and estimators.

N	$\varepsilon = 1$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-10}$	$\varepsilon = 2^{-15}$	$\varepsilon = 2^{-20}$	$\varepsilon = 2^{-25}$	$\varepsilon = 2^{-30}$
	Errors (odd rows) & Computational Rates (even rows)						
64	3.373e-4	3.723e-3	8.952e-3	8.973e-3	8.973e-3	8.973e-3	8.973e-3
	2.00	1.91	1.01	1.00	1.00	1.00	1.00
128	8.445e-5	9.935e-4	4.446e-3	4.484e-3	4.484e-3	4.484e-3	4.484e-3
	2.00	1.98	1.04	1.00	1.00	1.00	1.00
256	2.112e-5	2.523e-4	2.165e-3	2.236e-3	2.236e-3	2.236e-3	2.236e-3
	SECOND Estimator (odd rows) & Effectivity Indices (even rows)						
64	7.353e-3	1.204e-1	1.224e-1	1.230e-1	1.302e-1	1.302e-1	1.302e-1
	21.80	32.33	13.68	14.48	14.51	14.51	14.51
128	1.885e-3	3.212e-2	6.005e-2	6.621e-2	6.646e-2	6.647e-2	6.647e-2
	22.32	32.33	13.51	14.77	14.82	14.82	14.82
256	4.771e-4	8.268e-3	3.073e-2	3.328e-2	3.354e-2	3.354e-2	3.354e-2
	22.59	32.77	14.20	14.89	15.00	15.00	15.00

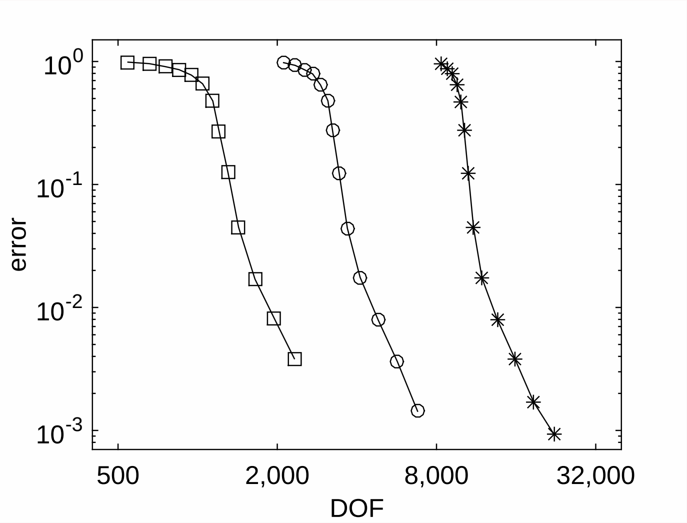
We considered one a-priori-chosen layer-adapted mesh of Bakhvalov type:

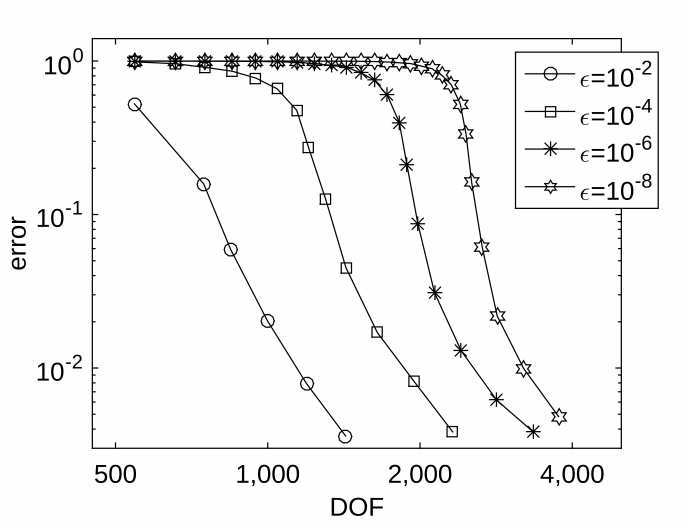


- The mesh is chosen so that the linear interpolation error $\|u - u^I\|_{\infty;\infty} \lesssim N^{-2}$.
- However, **as $\varepsilon \to 0$, the convergence rates deteriorate from 2 to 1**.
- E.g. for the final choice of ε and N, the aspect ratios of the mesh elements take values between 1 and 3.6e+8.
- Considering these variations, the SECOND estimator performs reasonably well and its efficitivity indices stabilize as $\varepsilon \to 0$.
- By contrast, the FIRST estimator is adequate for $\varepsilon \sim 1$, but its effectivity deteriorates in the singularly perturbed regime.

Simple 2d TEST problem: $-\varepsilon^2 \triangle u + u = F(x)$ in $\Omega = (0,1)^2$ with $\varepsilon^2 = 10^{-6}$,
$u = 4y\,(1-y)\,[1 - x^2 - (e^{-x/\varepsilon} - e^{-1/\varepsilon})/(1 - e^{-x/\varepsilon})]$

Maximum errors for $\varepsilon = 10^{-4}$ and initial DOF varied (left), and ε varied (right):





In each experiment, we started with a uniform mesh of right-angled triangles of diameter $H_T = 2^{-8}, 2^{-16}, 2^{-32}$, and aspect ratio $\frac{H_T}{h_T} = 2$. At each iteration, we marked for refinement the mesh elements responsible for at least 5% of the overall estimator $\mathcal{E}$, but no more than 15% of the elements. The marked elements were refined only in the x direction using a single or triple green refinement (depending on the orientation of the mesh element). Edge swapping was also employed to improve geometric properties of the mesh and/or possibly reduce $\max_{T\in\mathcal{T}}\{\mathrm{osc}(f_h^I;T)\}$.

Our estimators are also useful for a more challenging parabolic equations. Indeed, **plugging** them (as error estimators for elliptic reconstructions) into the **parabolic estimators** [Kopteva & Linß, SINUM, 2013] or [Demlow, Lakkis, Makridakis, SINUM, 2009, $-\triangle u = f$] yields a posteriori error estimates for the parabolic case.

- EXAMPLE [Fully Discrete Backward Euler]:

  With the elliptic estimator $\| u_h^{elliptic} - u^{elliptic} \|_\infty \leq \eta$

  one can **plug it** into a parabolic estimator for $\partial_t u - \varepsilon^2 \triangle u + f(x,t,u) = 0$...

$$
\begin{aligned}
\big\| u_h^m - u(\cdot, t_m) \big\|_{\infty,\Omega} \quad & \leq \kappa_0 \, e^{-C_f t_m} \big\| u_h^0 - \varphi \big\|_{\infty,\Omega} \\
& + (\kappa_1 \, \ell_m) \max_{j=1,\dots,m-1} \Big\{ \big\| u_h^j - u_h^{j-1} \big\|_{\infty,\Omega} + \eta^j \Big\} \\
& + 2\kappa_0 \, \big\| u_h^m - u_h^{m-1} \big\|_{\infty,\Omega} + (\kappa_0 + 1) \, \eta^m \\
& + \kappa_0 \, \sum_{j=1}^m \int_{t_{j-1}}^{t_j} e^{-C_f (t_m - s)} \, \big\| \vartheta(\cdot, s) \big\|_{\infty,\Omega} \, \mathrm{d}s
\end{aligned}
$$

(Here ϑ is essentially a data oscillation term.)

- **Convection-Diffusion**

— joint work with S. Franz and A. Demlow

— shape-regular meshes

— using S. Franz and N. Kopteva, Green's function estimates for a singularly perturbed convection-diffusion problem J. Differential Equations, 252 (2012)

- **Anisotropic mesh elements**

— more numerics (with more sophisticated adaptivity)

— 3d: flat and needle elements require different treatment

— the bounds in the non-singularly-perturbed case can be improved...

— other norms??

- N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes*, SINUM (2015).
- A. Demlow & N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed elliptic reaction-diffusion problems*, Numer. Math. (2015).
- N.Kopteva, *Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations*, Math. Comp. (2014).
- Kopteva & T. Linß, *Maximum norm a posteriori error estimation for parabolic problems using elliptic reconstructions*, SINUM (2013).

Final

Thank you!