Adaptive Discontinuous Galerkin Methods on Polytopic Meshes

Paul Houston

School of Mathematical Sciences, University of Nottingham, UK

AptoFEM

Joint work with Paola Antonietti (MOX, Milan), Andrea Cangiani (Leicester), Joe Collis (Nottingham), Peter Dong (Leicester), Manolis Georgoulis (Leicester) and Stefano Giani (Durham)

Outline

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

- Background
- FEMs on Polytopic Meshes
- Error Estimation
- Agglomeration-based Adaptivity
- Domain Decomposition Preconditioners
- Summary and Outlook

Background

Hackbusch & Sauter 1997→

- PDE problem: given $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ and $f \in \mathcal{H}$, find $u \in D(\mathcal{L})$ such that

$$\mathcal{L}u = f \quad \text{in } \Omega.$$

- Assume that Ω is *complicated* in the sense that it contains microstructures.
- FEM: given a mesh $\mathcal{T}_h$ of granularity h, find $u_h \in V_h(\mathcal{T}_h)$ such that

$$\mathcal{L}_h u_h = f_h.$$

- Standard element shapes: $\dim(V_h(\mathcal{T}_h)) \propto$ Complexity of Ω.

Meshing Complicated Geometries

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Hackbusch & Sauter 1997→

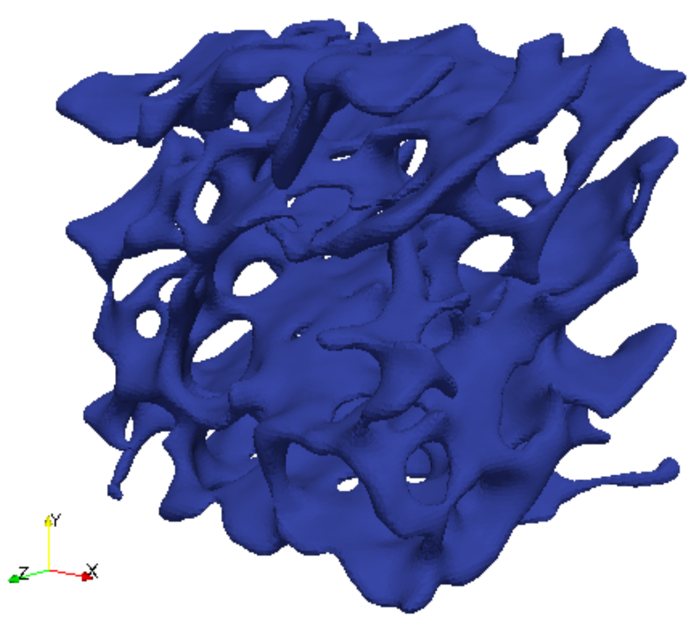
- PDE problem: given $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ and $f \in \mathcal{H}$, find $u \in D(\mathcal{L})$ such that

$$\mathcal{L}u = f \quad \text{in } \Omega.$$

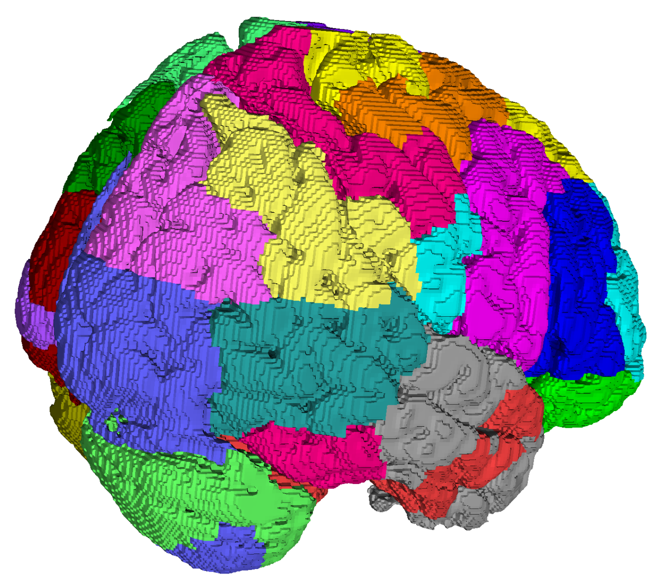
- Assume that Ω is *complicated* in the sense that it contains microstructures.
- FEM: given a mesh $\mathcal{T}_h$ of granularity h, find $u_h \in V_h(\mathcal{T}_h)$ such that

$$\mathcal{L}_h u_h = f_h.$$

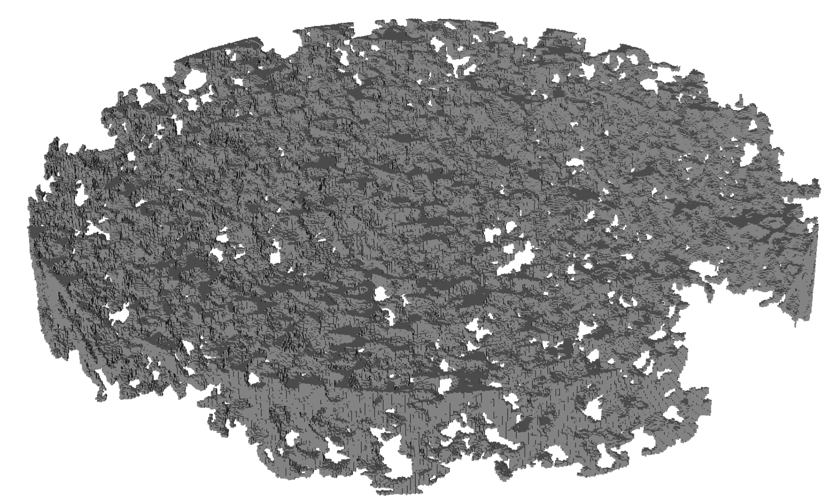
- Standard element shapes: $\dim(V_h(\mathcal{T}_h)) \propto$ Complexity of Ω.



1.2M Elements



1.6M Elements



15.8M Elements

Meshing Complicated Geometries

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Hackbusch & Sauter 1997→

- PDE problem: given $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ and $f \in \mathcal{H}$, find $u \in D(\mathcal{L})$ such that

$$\mathcal{L}u = f \quad \text{in } \Omega.$$

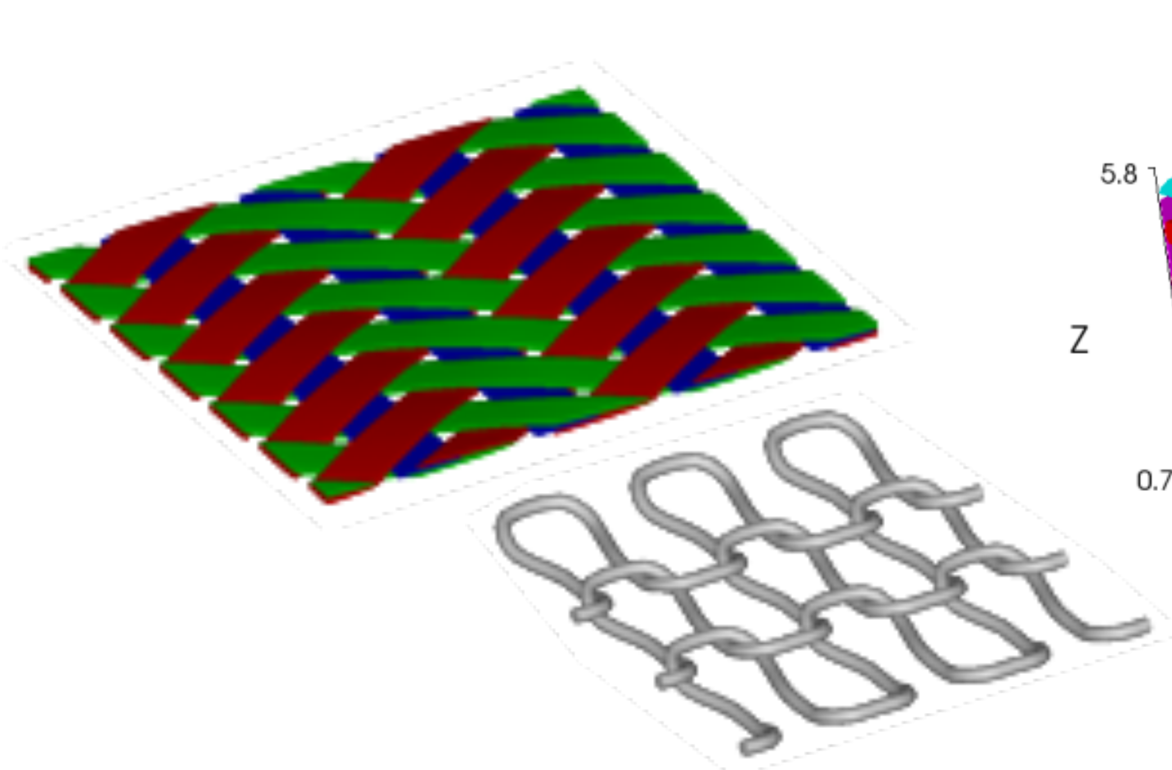
- Assume that Ω is *complicated* in the sense that it contains microstructures.
- FEM: given a mesh $\mathcal{T}_h$ of granularity h, find $u_h \in V_h(\mathcal{T}_h)$ such that

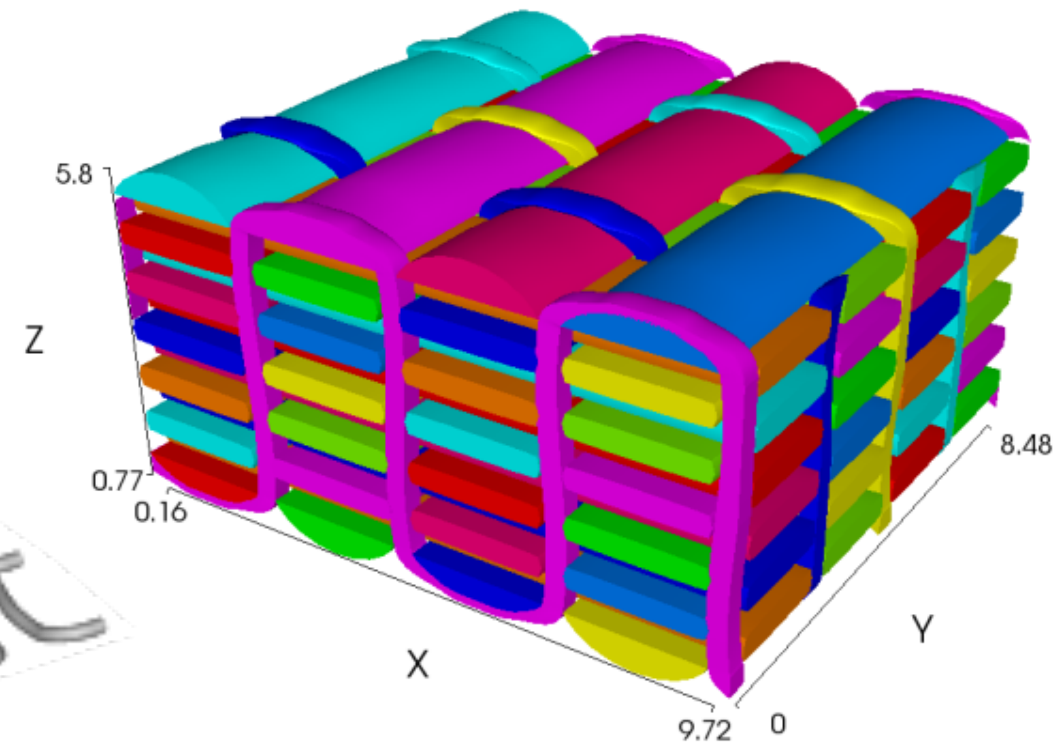
$$\mathcal{L}_h u_h = f_h.$$

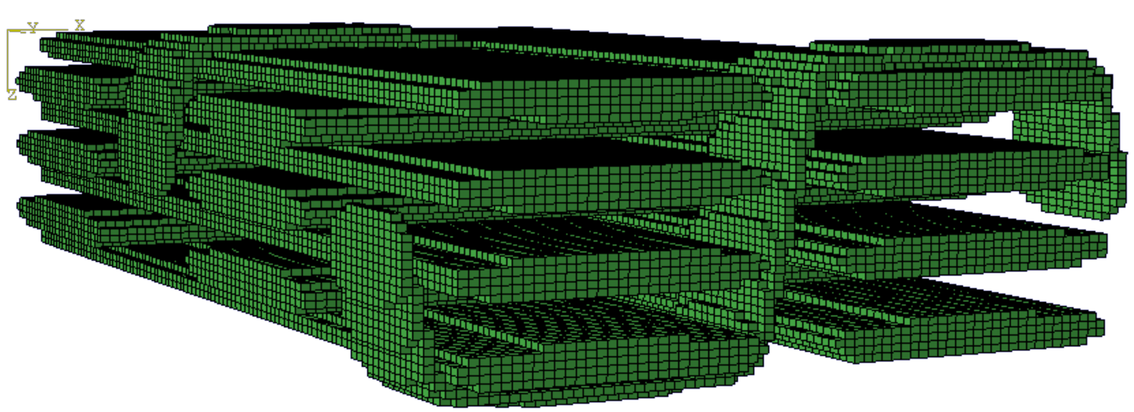
- Standard element shapes: $\dim(V_h(\mathcal{T}_h)) \propto$ Complexity of Ω.

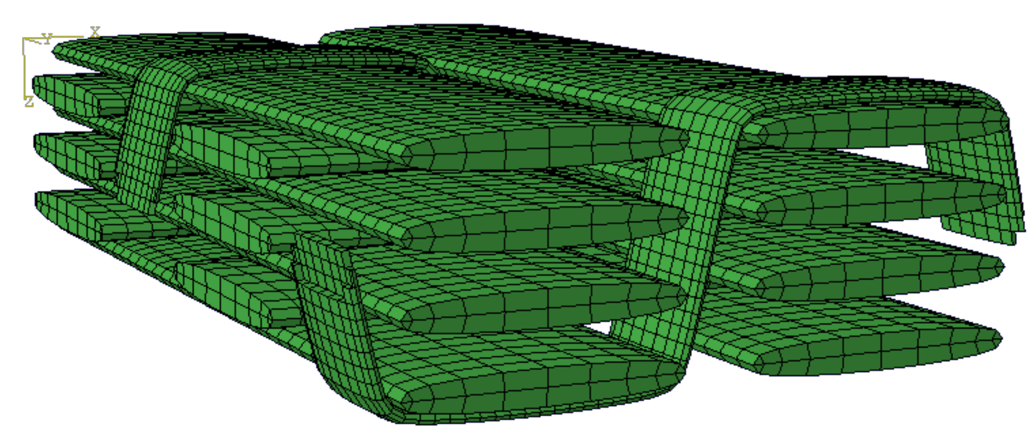
  - ☆ Number of degrees of freedom is *independent* of the domain;
  - ☆ Coarse approximations may be computed with engineering accuracy;
  - ☆ Adaptivity is focused on resolving *important features* of the solution;
  - ☆ Method naturally admits high-order polynomial orders;
  - ☆ May be exploited as coarse level solvers with multilevel preconditioners.

Textiles/Composites



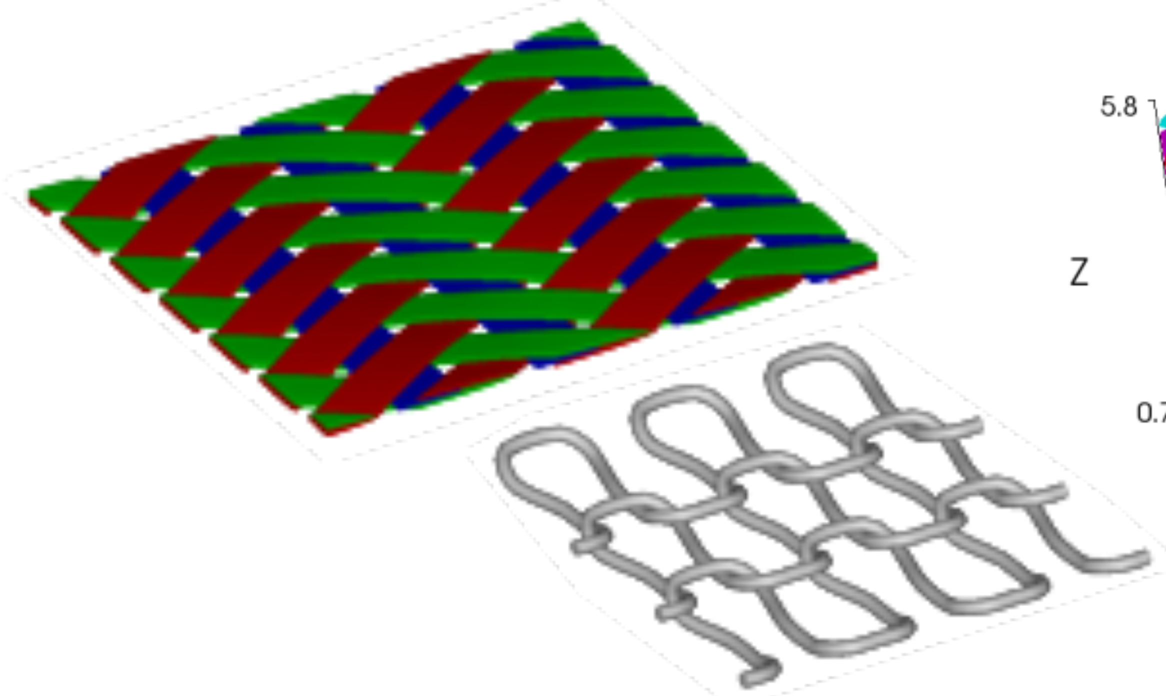


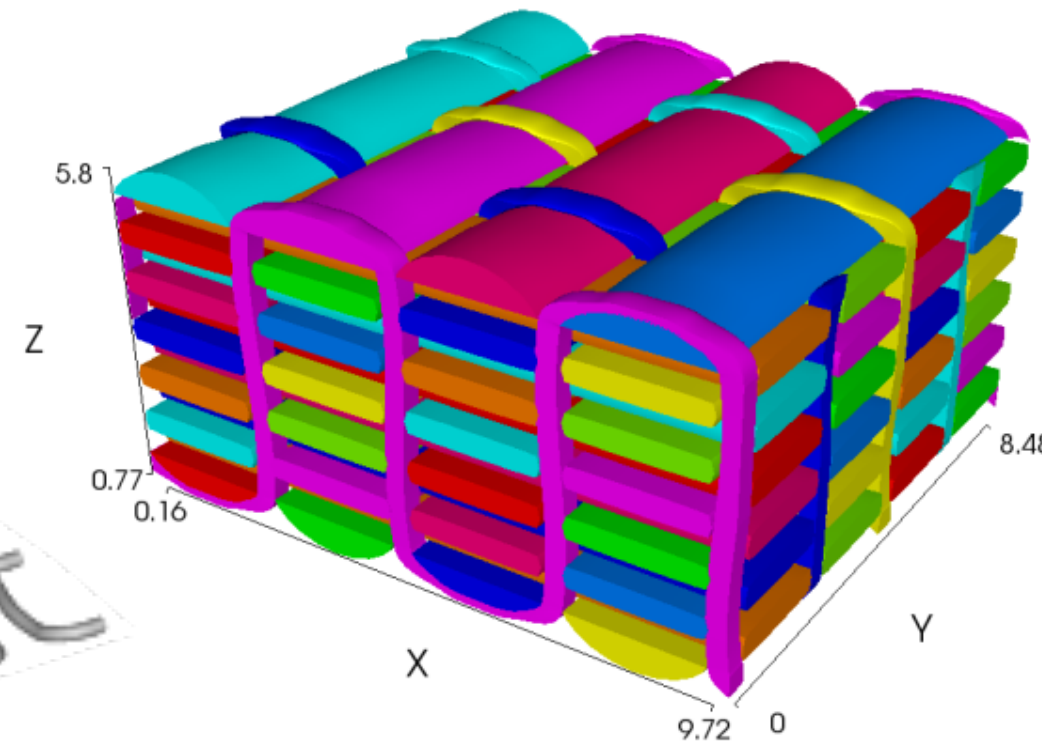


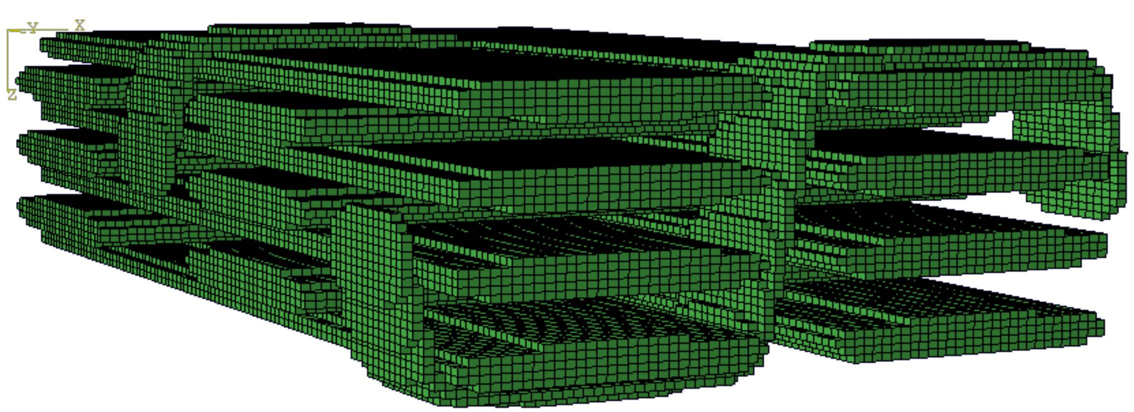


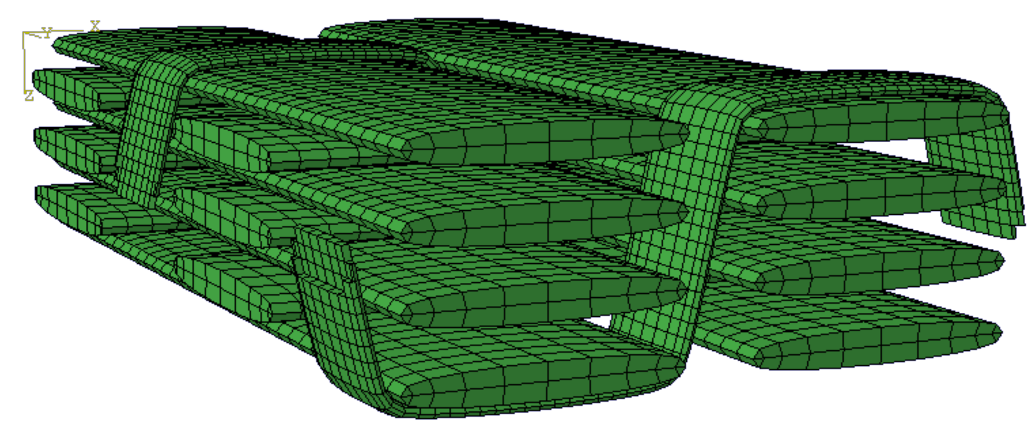
Joint work with Louise Brown, Mikhail Matveev, and Xuesen Zeng (University of Nottingham)

Textiles/Composites









Joint work with Louise Brown, Mikhail Matveev, and Xuesen Zeng (University of Nottingham)

- Other applications include: Gearbox design (Romax), fluid structure interaction, geophysical problems, for example, earth-quake engineering and flows in fractured porous media.

FEMs on Polytopic Meshes

FEMs on Polygonal/Polyhedral Meshes

- Polygonal Finite Element Methods.
  Sukumar & Tabarraei 2004, 2007
- Extended/Generalised FEMs (Partition of Unity).
  Duarte & Oden 1996, Melenk & Babuska 1996, Moes, Dolbow, & Belytschko 1999, Daux, Moes, Dolbow, Sukumar, & Belytschko 2000, Sukumar, Moes, Moran, & Belytschko 2000, Belytschko, Moes, Usui, & Parimi 2001, Gerstenberger & Wall 2008, Bechet, Moes, & Wohlmuth 2009, Belytschko, Gracie, & Ventura 2009, Jaroslav & Renard 2009, Fries & Belytschko 2010, Shahmiri, Gerstenberger, & Wall 2011, …
- Virtual Element Method.
  Beirao da Veiga, Brezzi, Cangiani, Manzini, Marini, & Russo 2013
- Mimetic Finite Difference Method.
  Brezzi, Lipnikov, & Shashkov 2005, Brezzi, Lipnikov, & Simoncini 2005, Brezzi, Buffa, & Lipnikov 2009, Cangiani, Manzini, Russo 2009, Beirao da Veiga, Droniou, & Manzini 2011, Beirao da Veiga, Lipnikov & Manzini 2011, Beirao da Veiga & Manzini 2013,…
- Hybrid High-Order Methods.
  Di Pietro & Ern 2015, Di Pietro, Ern, & Lemaire 2015.
- Composite Finite Element Methods.
  Shortley & Weller 1938, Hackbusch & Sauter 1997→, Rech, Sauter, & Smolianski 2006, Antonietti, Giani, & H. 2012, 2013,…
- Agglomerated Finite Element Methods.
  DGFEM: Bassi, Botti, Colombo, Di Pietro, & Tesini 2012, Bassi, Botti & Colombo 2013.

FEMs on Polygonal/Polyhedral Meshes

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

- Polygonal Finite Element Methods.

  Sukumar & Tabarraei 2004, 2007

- Extended/Generalised FEMs (Partition of Unity).

  Duarte & Oden 1996, Melenk & Babuska 1996, Moes, Dolbow, & Belytschko 1999, Daux, Moes, Dolbow, Sukumar, & Belytschko 2000, Sukumar, Moes, Moran, & Belytschko 2000, Belytschko, Moes, Usui, & Parimi 2001, Gerstenberger & Wall 2008, Bechet, Moes, & Wohlmuth 2009, Belytschko, Gracie, & Ventura 2009, Jaroslav & Renard 2009, Fries & Belytschko 2010, Shahmiri, Gerstenberger, & Wall 2011, ...

- Virtual Element Method.

  Beirao da Veiga, Brezzi, Cangiani, Manzini, Marini, & Russo 2013

- Mimetic Finite Difference Method.

  Brezzi, Lipnikov, & Shashkov 2005, Brezzi, Lipnikov, & Simoncini 2005, Brezzi, Buffa, & Lipnikov 2009, Cangiani, Manzini, Russo 2009, Beirao da Veiga, Droniou, & Manzini 2011, Beirao da Veiga, Lipnikov & Manzini 2011, Beirao da Veiga & Manzini 2013,...

- Hybrid High-Order Methods.

  Di Pietro & Ern 2015, Di Pietro, Ern, & Lemaire 2015.

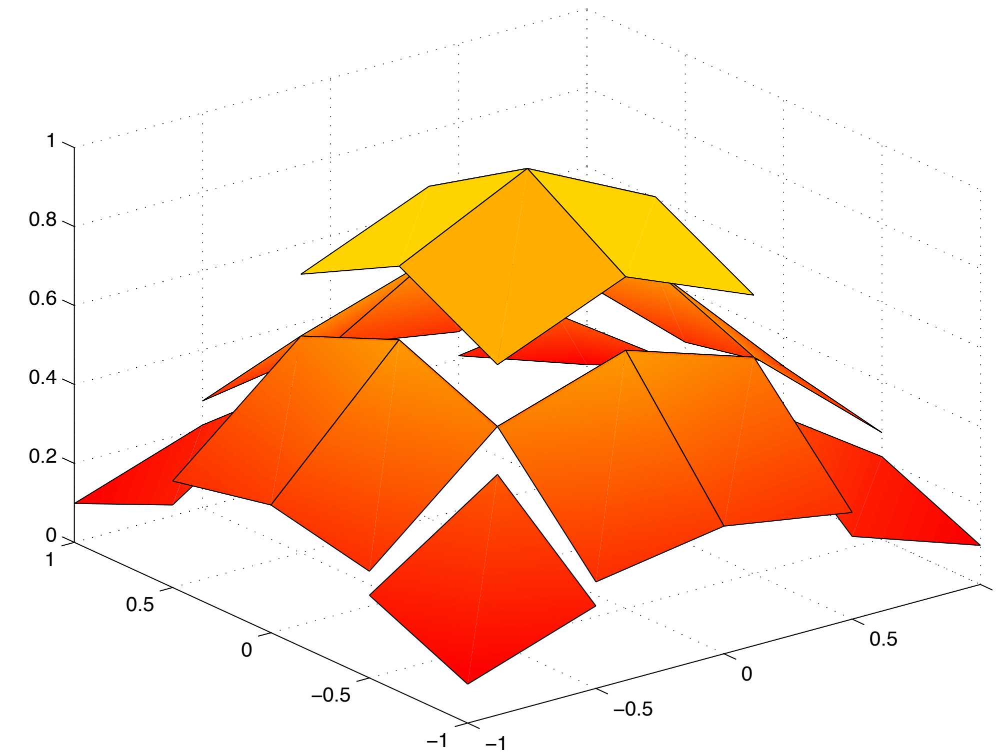
- Composite Finite Element Methods.

  Shortley & Weller 1938, Hackbusch & Sauter 1997→, Rech, Sauter, & Smolianski 2006, Antonietti, Giani, & H. 2012, 2013,...

- Agglomerated Finite Element Methods.

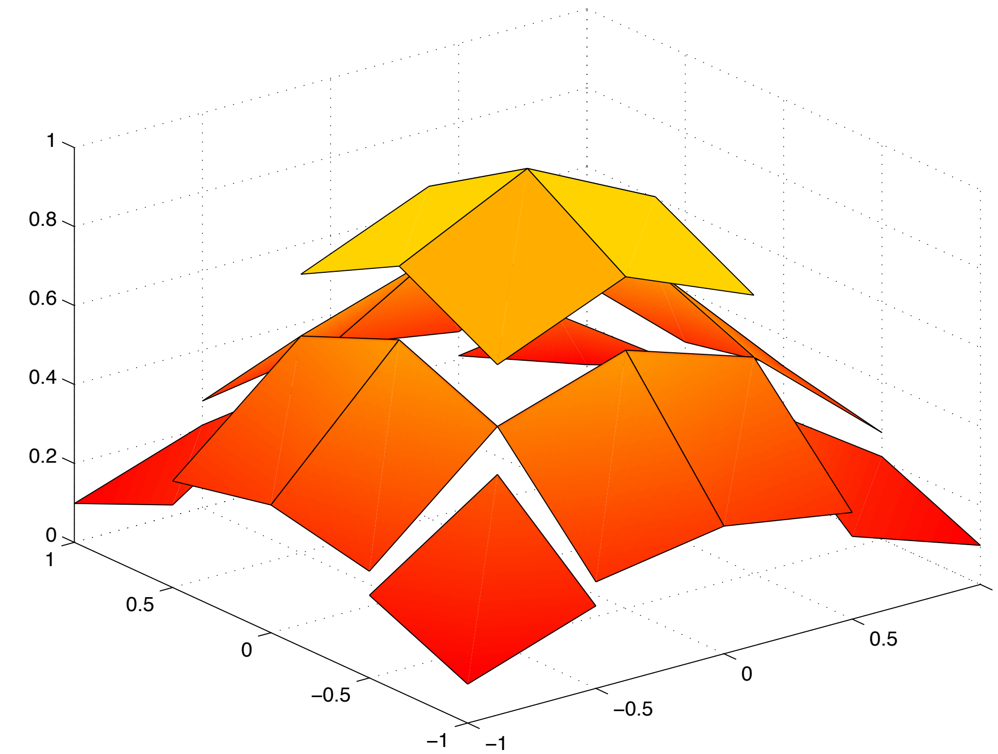
  DGFEM: Bassi, Botti, Colombo, Di Pietro, & Tesini 2012, Bassi, Botti & Colombo 2013.

- Method Construction:
  - Local (elementwise) weak formulation.
  - Weak Imposition of the boundary conditions (Numerical fluxes).
  - Gives rise to a globally coupled system of equations.



- Method Construction:
  - Local (elementwise) weak formulation.
  - Weak Imposition of the boundary conditions (Numerical fluxes).
  - Gives rise to a globally coupled system of equations.
- Elliptic PDEs

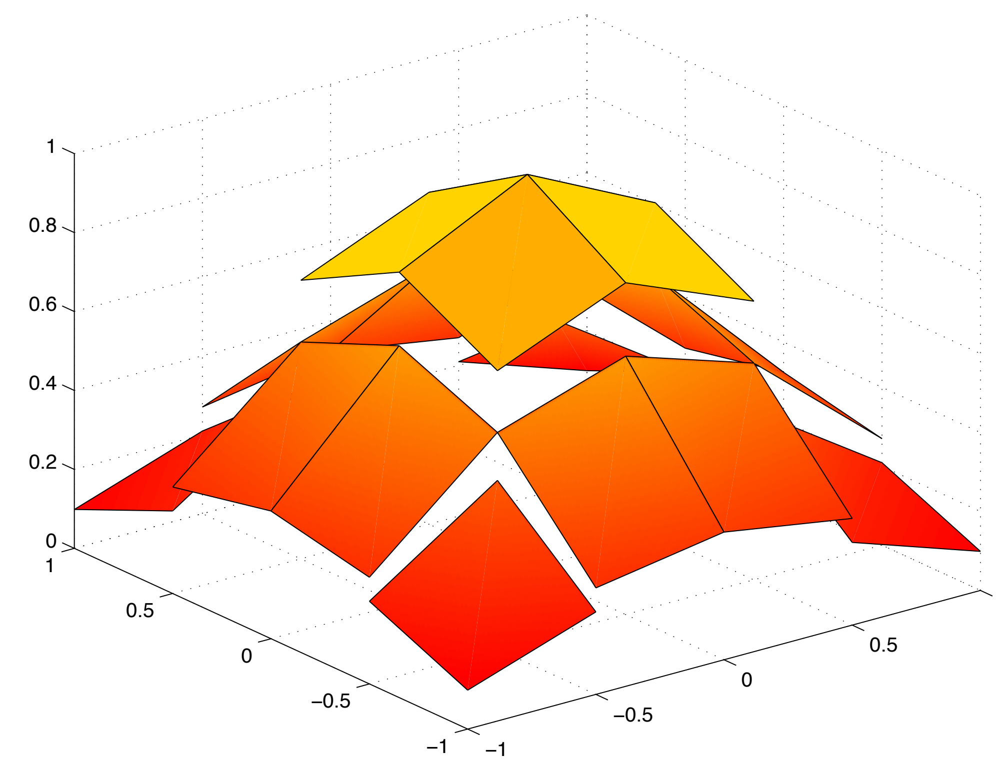
  Pian 1965, Nitsche 1971, Wheeler 1978, Arnold 1982, …



- Method Construction:
  - Local (elementwise) weak formulation.
  - Weak Imposition of the boundary conditions (Numerical fluxes).
  - Gives rise to a globally coupled system of equations.
- Elliptic PDEs

  Pian 1965, Nitsche 1971, Wheeler 1978, Arnold 1982, …
- Hyperbolic PDEs

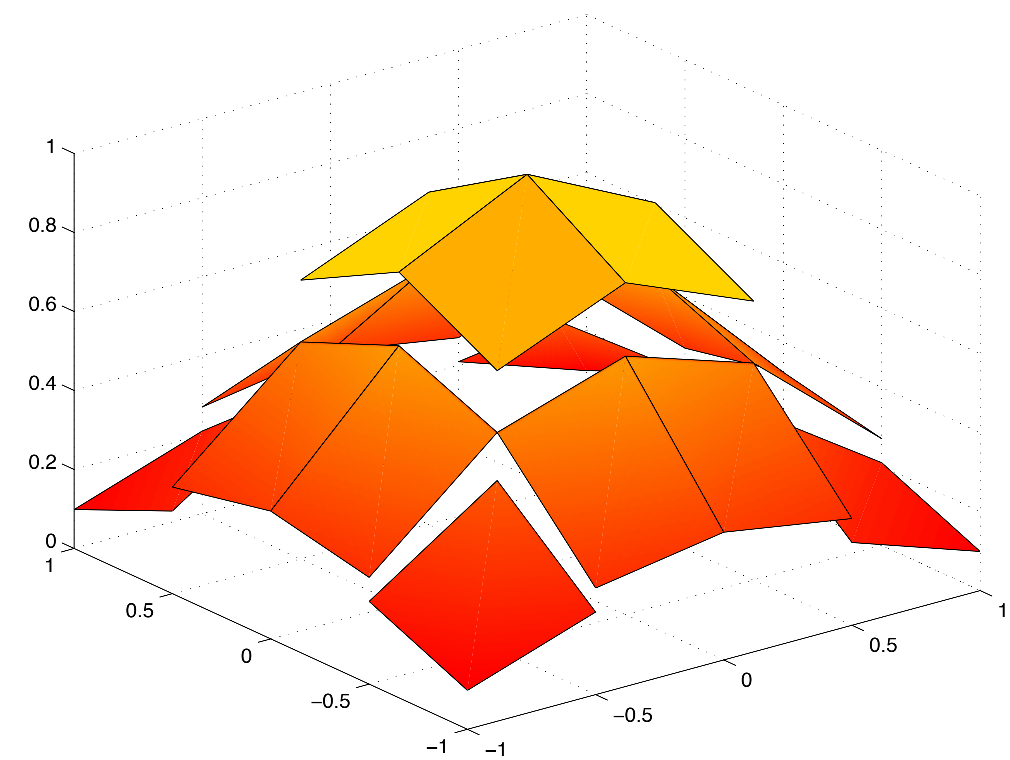
  Reed & Hill 1973, Lesaint & Raviart 1974, Johnson, Navert & Pitkaranta 1984



- Method Construction:
  - Local (elementwise) weak formulation.
  - Weak Imposition of the boundary conditions (Numerical fluxes).
  - Gives rise to a globally coupled system of equations.
- Elliptic PDEs

  Pian 1965, Nitsche 1971, Wheeler 1978, Arnold 1982, …
- Hyperbolic PDEs

  Reed & Hill 1973, Lesaint & Raviart 1974, Johnson, Navert & Pitkaranta 1984



- Applications

  Linear elliptic/parabolic/hyperbolic PDEs, Fokker-Planck equations, Incompressible/Compressible fluid flows, Turbulent flows, Non-Newtonian flows, Time and frequency domain Maxwell's equations, Acoustics, MHD, Fully nonlinear PDEs.

Discontinuous Galerkin Methods

- ✓ Robustness/stability;
- ✓ Locally conservative;
- ✓ *Ease of implementation;*
- ✓ Highly parallelizable;
- ✓ Flexible mesh design (hybrid grids, non-matching grids, non-uniform/anisotropic polynomial degrees);
- ✓ Wider choice of stable FE spaces for mixed problems;
- ✓ Unified treatment of a wide range of PDEs;

Discontinuous Galerkin Methods

- ✓ Robustness/stability;
- ✓ Locally conservative;
- ✓ *Ease of implementation;*
- ✓ Highly parallelizable;
- ✓ Flexible mesh design (hybrid grids, non-matching grids, non-uniform/anisotropic polynomial degrees);
- ✓ Wider choice of stable FE spaces for mixed problems;
- ✓ Unified treatment of a wide range of PDEs;

★ Computational overhead/efficiency (increase in DoFs);

Discontinuous Galerkin Methods

- ✓ Robustness/stability;
- ✓ Locally conservative;
- ✓ *Ease of implementation;*
- ✓ Highly parallelizable;
- ✓ Flexible mesh design (hybrid grids, non-matching grids, non-uniform/anisotropic polynomial degrees);
- ✓ Wider choice of stable FE spaces for mixed problems;
- ✓ Unified treatment of a wide range of PDEs;
- ✓ Convergence of the method is *independent* of the element shape;
  - ➡ Polynomial bases may be defined in the physical frame, without the need to map from a reference element.

    (See Bassi, Botti, Colombo, Di Pietro, & Tesini 2012)
- ★ Computational overhead/efficiency (increase in DoFs);

PDE Problem

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Poisson's Equation

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and $f \in L_2(\Omega)$: find u such that

$$-\Delta u = f \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega.$$

PDE Problem

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Poisson's Equation

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and $f \in L_2(\Omega)$: find u such that

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

- Assume that Ω is *complicated* in the sense that it contains microstructures.

PDE Problem

Poisson's Equation

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and $f \in L_2(\Omega)$: find u such that

$$-\Delta u = f \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega.$$

- Assume that Ω is *complicated* in the sense that it contains microstructures.

Theorem

There exists a linear extension operator $\mathfrak{E} : H^s(\Omega) \to H^s(\mathbb{R}^d)$, $s \in \mathbb{N}_0$, such that $\mathfrak{E}v|_\Omega = v$ and

$$\|\mathfrak{E}v\|_{H^s(\mathbb{R}^d)} \leq C\|v\|_{H^s(\Omega)}.$$

See Stein 1970, Sauter & Warnke 1999.

Mesh Construction

Mesh Construction

- Exploit coarse meshes consisting of polygonal/polyhedral elements.

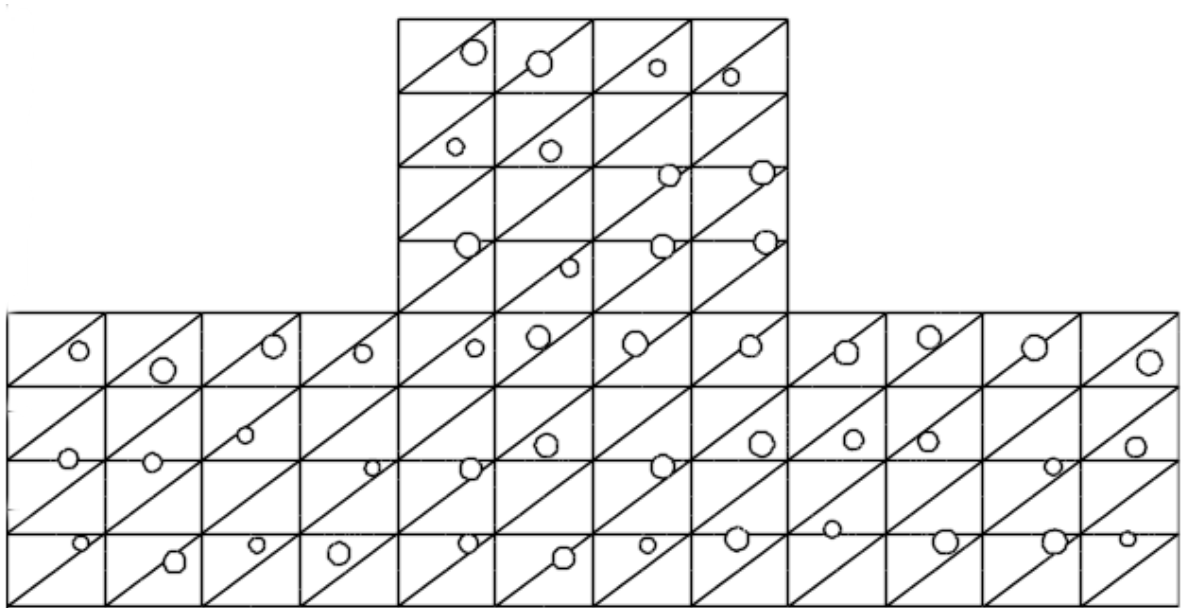
Mesh Construction

- Exploit coarse meshes consisting of polygonal/polyhedral elements.
  - Polygonal mesh generator, e.g., `Polymesher`, cf. Talischi *et al.* 2012.

Mesh Construction

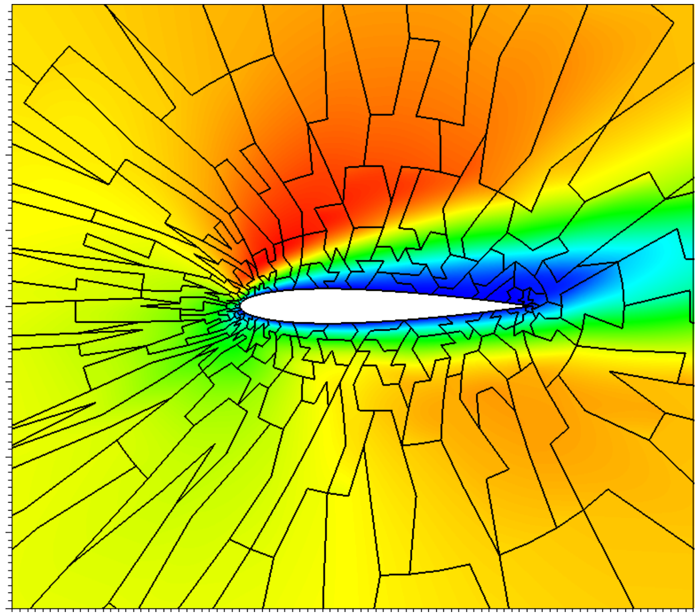
- Exploit coarse meshes consisting of polygonal/polyhedral elements.
  - Polygonal mesh generator, e.g., `Polymesher`, cf. Talischi *et al.* 2012.
  - Agglomeration of fine geometry-conforming mesh:

Overlapping Refined Mesh



Hackbusch & Sauter 1997, Antonietti, Giani, & H. 2012

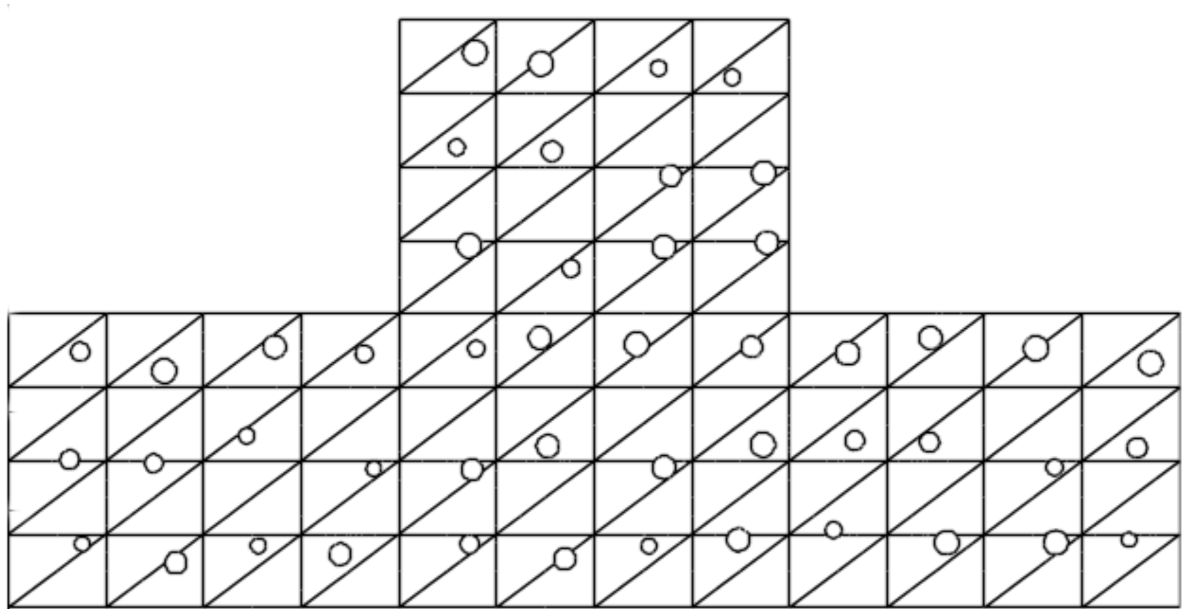
Graph Partitioning, e.g., `METIS`



Mesh Construction

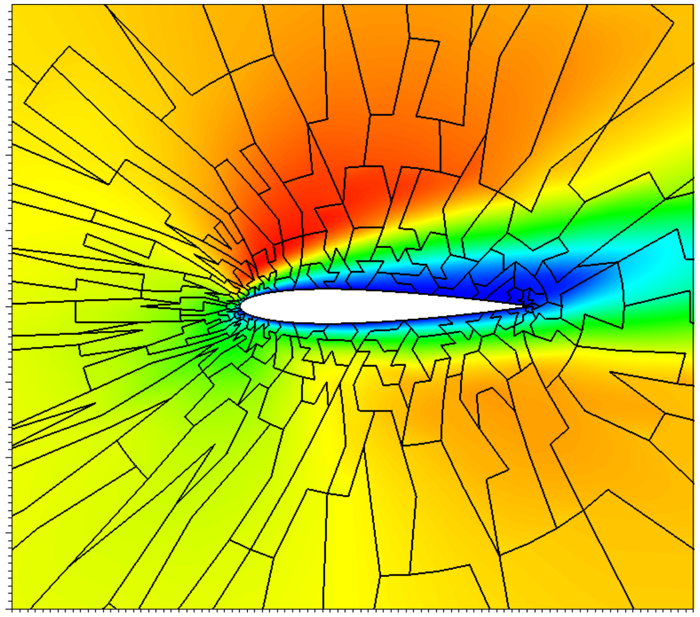
- Exploit coarse meshes consisting of polygonal/polyhedral elements.
  - Polygonal mesh generator, e.g., `Polymesher`, cf. Talischi *et al.* 2012.
  - Agglomeration of fine geometry-conforming mesh:

Overlapping Refined Mesh



Hackbusch & Sauter 1997, Antonietti, Giani, & H. 2012

Graph Partitioning, e.g., `METIS`



- This allows for the construction of very coarse finite element meshes, even on complicated domains containing microstructures.
- Mesh can then be automatically refined on the basis of solution accuracy.

DG Finite Element Spaces

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

We set

$$V(\mathcal{T}_{\mathtt{CFE}}, \mathbf{p}) = \{u \in L_2(\Omega) : u|_\kappa \in \mathcal{P}_{p_\kappa}(\kappa)\ \forall \kappa \in \mathcal{T}_{\mathtt{CFE}}\},$$

where $\mathcal{P}_p(\kappa)$ denotes the set of polynomials of degree at most $p \geq 1$ over κ.

➡ Polynomial bases are defined in the physical space, *without* any mappings.

DG Finite Element Spaces

We set

$$V(\mathcal{T}_{\mathtt{CFE}}, \mathbf{p}) = \{u \in L_2(\Omega) : u|_\kappa \in \mathcal{P}_{p_\kappa}(\kappa)\ \forall \kappa \in \mathcal{T}_{\mathtt{CFE}}\},$$

where $\mathcal{P}_p(\kappa)$ denotes the set of polynomials of degree at most $p \geq 1$ over κ.

➡ Polynomial bases are defined in the physical space, *without* any mappings.

Mesh Assumptions

- $\mathcal{F}(\mathcal{T}_{\mathtt{CFE}}) = \mathcal{F}^{\mathcal{I}}_{\mathtt{CFE}} \cup \mathcal{F}^{\mathcal{B}}_{\mathtt{CFE}}$ denotes the set of all faces in the mesh $\mathcal{T}_{\mathtt{CFE}}$.

(A1) For all elements $\kappa \in \mathcal{T}_{\mathtt{CFE}}$, we require

$$\max_{\kappa \in \mathcal{T}_{\mathtt{CFE}}} \text{card}\left\{F \in \mathcal{F}^{\mathcal{I}}_{\mathtt{CFE}} \cup \mathcal{F}^{\mathcal{B}}_{\mathtt{CFE}} : F \subset \partial\kappa\right\} \leq C_F \quad \text{(uniformly)}.$$

(A2) The polynomial degree vector $\mathbf{p}$ is of bounded local variation.

hp-DGFEM (based on Symmetric Interior Penalty Method- SIPG)

Find $u_h \in V(\mathcal{T}_{\mathtt{CFE}}, \mathbf{p})$ such that

$$B_{\mathtt{DG}}(u_h, v) = F_h(v)$$

for all $v \in V(\mathcal{T}_{\mathtt{CFE}}, \mathbf{p})$, where

$$\begin{aligned} B_{\mathtt{DG}}(u, v) &= \sum_{\kappa \in \mathcal{T}_{\mathtt{CFE}}} \int_\kappa \nabla u \cdot \nabla v \, dx + \sum_{F \in \mathcal{F}^{\mathcal{I}}_{\mathtt{CFE}} \cup \mathcal{F}^{\mathcal{B}}_{\mathtt{CFE}}} \int_F \sigma \, [\![u]\!] \cdot [\![v]\!] \, ds \\ &- \sum_{F \in \mathcal{F}^{\mathcal{I}}_{\mathtt{CFE}} \cup \mathcal{F}^{\mathcal{B}}_{\mathtt{CFE}}} \int_F \left(\{\!\{\nabla_h v\}\!\} \cdot [\![u]\!] + \{\!\{\nabla_h u\}\!\} \cdot [\![v]\!] \right) ds, \\ F_h(v) &= \int_\Omega f v \, dx. \end{aligned}$$

$\{\!\{\cdot\}\!\}$: Average Operator $\qquad [\![\cdot]\!]$: Jump Operator

hp-DGFEM (based on Symmetric Interior Penalty Method- SIPG)

Find $u_h \in V(\mathcal{T}_{\mathrm{CFE}}, \mathbf{p})$ such that

$$B_{\mathrm{DG}}(u_h, v) = F_h(v)$$

for all $v \in V(\mathcal{T}_{\mathrm{CFE}}, \mathbf{p})$, where

$$\begin{aligned} B_{\mathrm{DG}}(u, v) &= \sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \int_\kappa \nabla u \cdot \nabla v \, dx + \sum_{F \in \mathcal{F}^{\mathcal{I}}_{\mathrm{CFE}} \cup \mathcal{F}^{\mathcal{B}}_{\mathrm{CFE}}} \int_F \sigma \, [\![u]\!] \cdot [\![v]\!] \, ds \\ &\quad - \sum_{F \in \mathcal{F}^{\mathcal{I}}_{\mathrm{CFE}} \cup \mathcal{F}^{\mathcal{B}}_{\mathrm{CFE}}} \int_F \left(\{\!\{\nabla_h v\}\!\} \cdot [\![u]\!] + \{\!\{\nabla_h u\}\!\} \cdot [\![v]\!] \right) ds, \\ F_h(v) &= \int_\Omega f v \, dx. \end{aligned}$$

Stabilisation

Symmetry

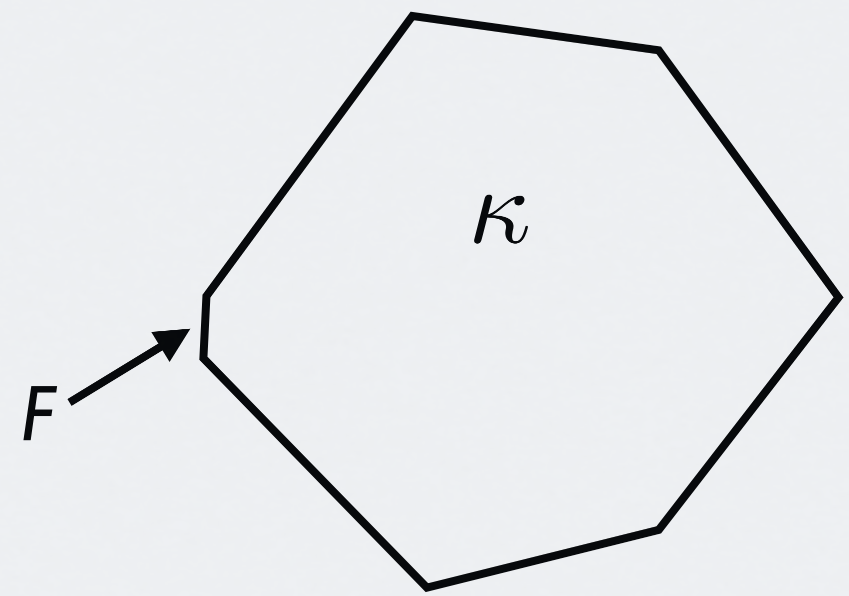
Consistency

$\{\!\{\cdot\}\!\}$: Average Operator $\quad$ $[\![\cdot]\!]$: Jump Operator

Error Estimation

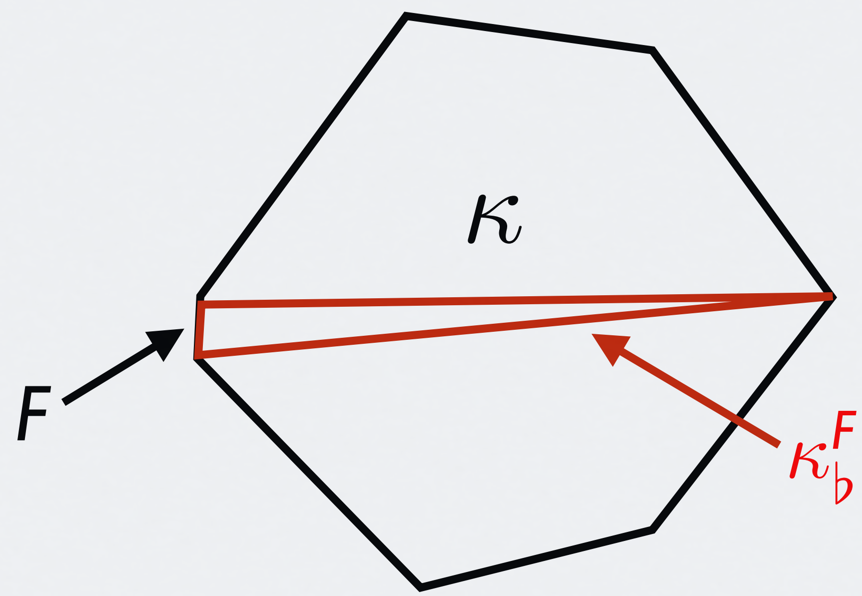
Face/Edge Degeneration

- $\text{meas}(F) \ll \text{meas}(\kappa)$



Face/Edge Degeneration

- $\text{meas}(F) \ll \text{meas}(\kappa)$



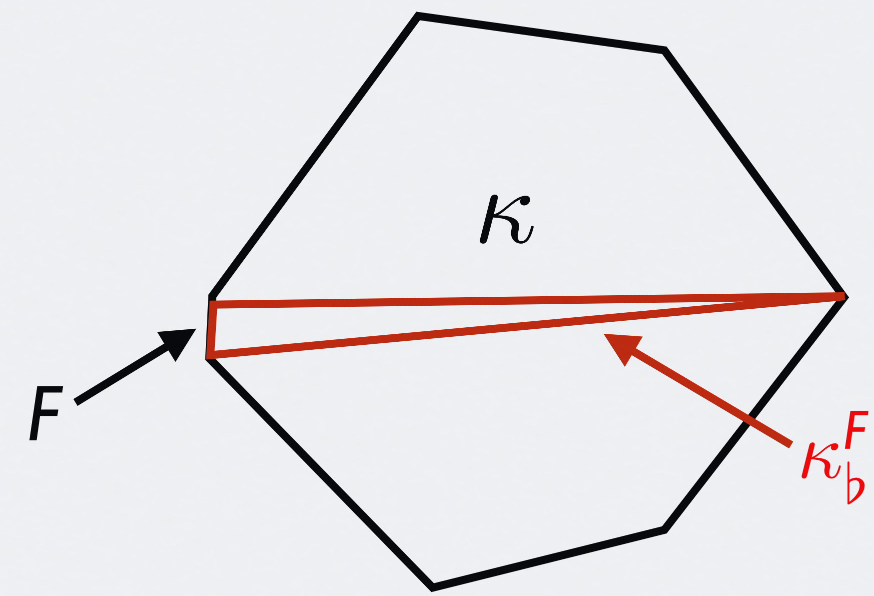
Inverse Estimate

Given $v \in \mathcal{P}_p(\kappa)$, we have the inverse estimate

$$\|v\|_{L^2(F)}^2 \leq C_{\mathrm{inv}} \frac{p^2 |F|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|} \|v\|_{L^2(\kappa)}^2.$$

Face/Edge Degeneration

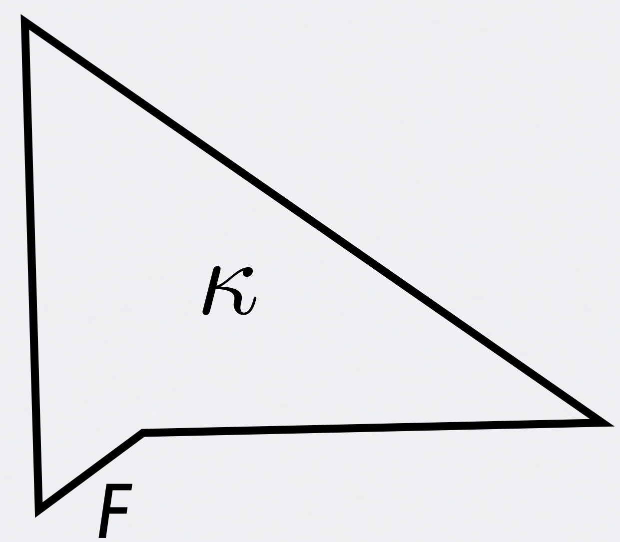
- $\text{meas}(F) \ll \text{meas}(\kappa)$



Inverse Estimate

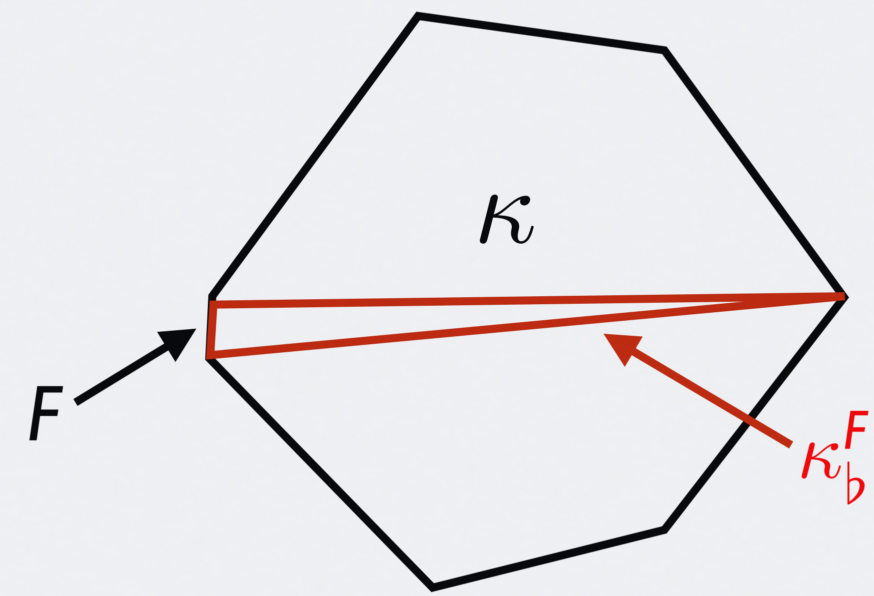
Given $v \in \mathcal{P}_p(\kappa)$, we have the inverse estimate

$$\|v\|_{L^2(F)}^2 \leq C_{\text{inv}} \frac{p^2 |F|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|} \|v\|_{L^2(\kappa)}^2.$$



Face/Edge Degeneration

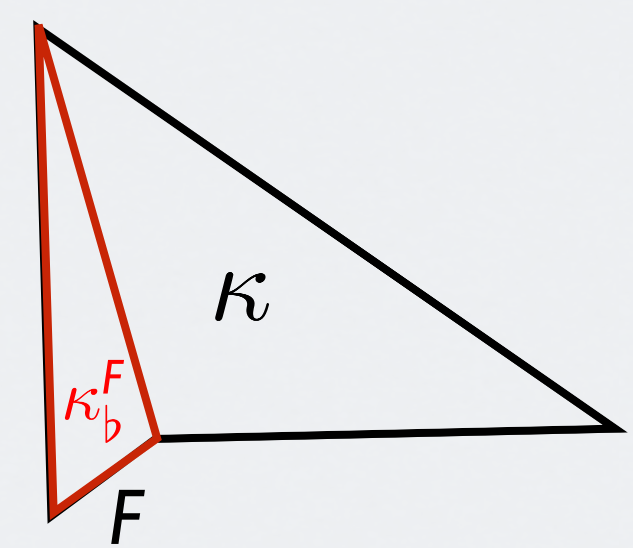
- $\text{meas}(F) \ll \text{meas}(\kappa)$



Inverse Estimate

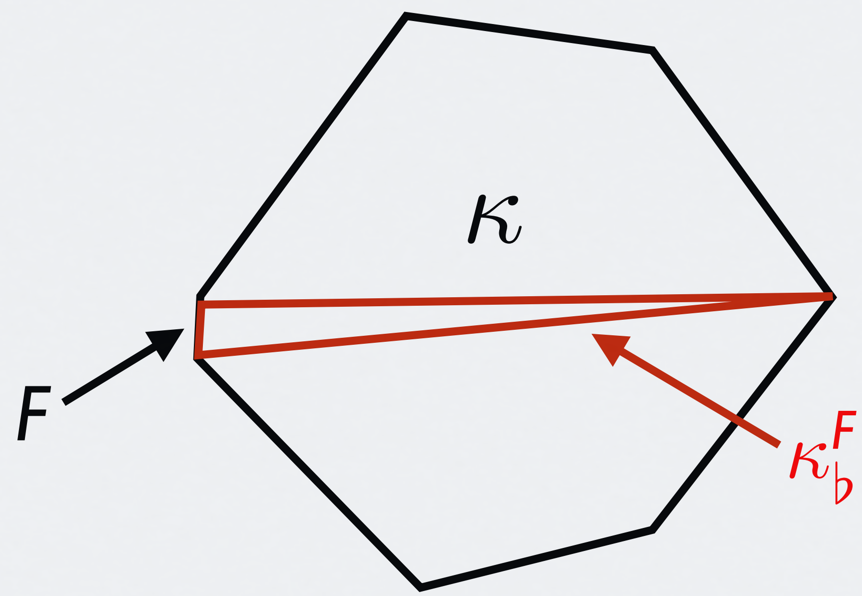
Given $v \in \mathcal{P}_p(\kappa)$, we have the inverse estimate

$$\|v\|_{L^2(F)}^2 \leq C_{\text{inv}} \frac{p^2|F|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|} \|v\|_{L^2(\kappa)}^2.$$



Face/edge Degeneration

- $\text{meas}(F) \ll \text{meas}(\kappa)$



Inverse Estimate

Given $v \in \mathcal{P}_p(\kappa)$, we have the inverse estimate

$$\|v\|^2_{L^2(F)} \leq C_{\text{inv}} \min\left\{ \frac{|\kappa|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, p^{2d} \right\} \frac{p^2 |F|}{|\kappa|} \|v\|^2_{L^2(\kappa)}.$$

Proof: Exploit an inverse inequality in L^∞, together with results from Georgoulis 2008.

DG-Norm

$$|||\, v \,|||_{\mathrm{DG}}^2 = \sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \|\nabla v\|_{L_2(\kappa)}^2 + \sum_{F \in \mathcal{F}_{\mathrm{CFE}}^{\mathcal{I}} \cup \mathcal{F}_{\mathrm{CFE}}^{\mathcal{B}}} \|\sigma^{1/2} [\![v]\!] \|_{L_2(F)}^2 .$$

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

DG-Norm

$$|||v|||_{\mathrm{DG}}^2 = \sum_{\kappa\in\mathcal{T}_{\mathrm{CFE}}} \|\nabla v\|_{L_2(\kappa)}^2 + \sum_{F\in\mathcal{F}_{\mathrm{CFE}}^{\mathcal{I}}\cup\mathcal{F}_{\mathrm{CFE}}^{\mathcal{B}}} \|\sigma^{1/2}[\![v]\!]\|_{L_2(F)}^2.$$

Interior Penalty Parameter

$$\sigma := \gamma\, C_{\mathrm{inv}} \max_{\kappa\in\{\kappa^+,\kappa^-\}} \left\{ \min\left\{ \frac{|\kappa|}{\sup_{\kappa_\flat^F\subset\kappa}|\kappa_\flat^F|}, p_\kappa^{2d} \right\} \frac{p_\kappa^2|F|}{|\kappa|} \right\}, \quad F = \kappa^+\cap\kappa^-.$$

DG-Norm

$$|||v|||_{DG}^2 = \sum_{\kappa \in \mathcal{T}_{\mathtt{CFE}}} \|\nabla v\|_{L_2(\kappa)}^2 + \sum_{F \in \mathcal{F}_{\mathtt{CFE}}^{\mathcal{I}} \cup \mathcal{F}_{\mathtt{CFE}}^{\mathcal{B}}} \|\sigma^{1/2} [\![v]\!]\|_{L_2(F)}^2.$$

Interior Penalty Parameter

$$\sigma := \gamma\, C_{\mathrm{inv}} \max_{\kappa \in \{\kappa^+, \kappa^-\}} \left\{ \min \left\{ \frac{|\kappa|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, p_\kappa^{2d} \right\} \frac{p_\kappa^2 |F|}{|\kappa|} \right\}, \quad F = \kappa^+ \cap \kappa^-.$$

Lemma (Coercivity & Continuity)

For $\gamma > \gamma_{\mathrm{min}}$, we have

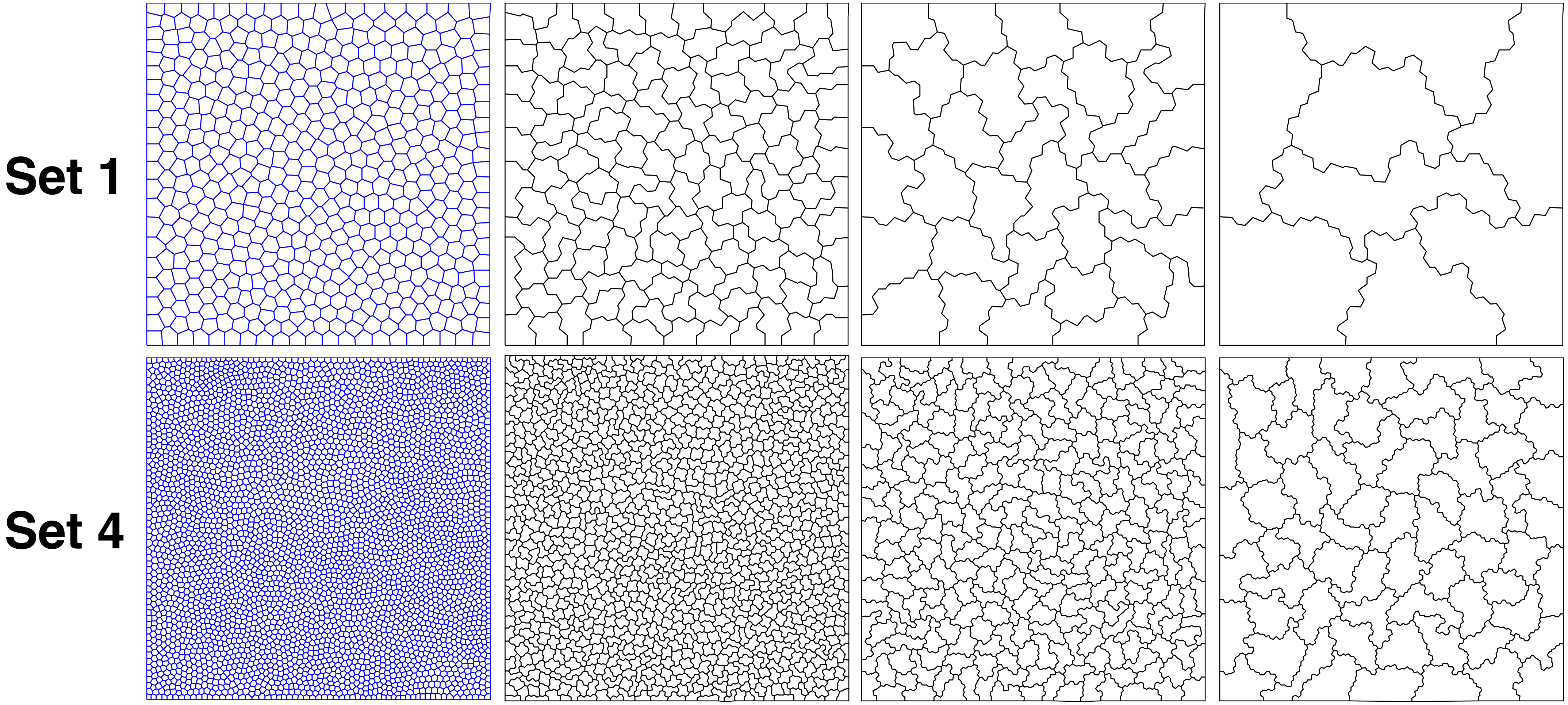
$$B_{\mathtt{DG}}(v, v) \quad \geq \quad C_{\mathrm{coer}} |||v|||_{\mathtt{DG}}^2 \quad \text{for all } v \in V(\mathcal{T}_{\mathtt{CFE}}, \mathbf{p}),$$

and

$$B_{\mathtt{DG}}(v, w) \quad \leq \quad C_{\mathrm{cont}} |||v|||_{\mathtt{DG}} |||w|||_{\mathtt{DG}} \quad \text{for all } v, w \in V(\mathcal{T}_{\mathtt{CFE}}, \mathbf{p}).$$

Stability of the DGFEM

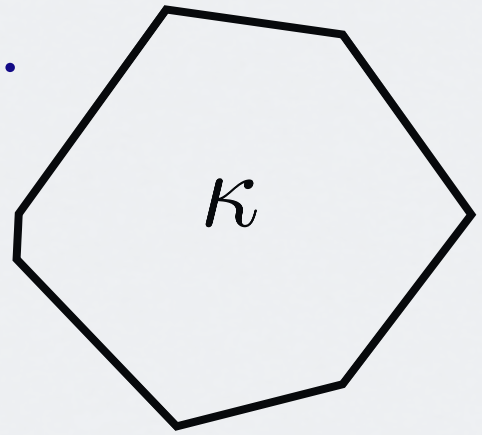
	Set 1	Set 2	Set 3	Set 4
Mesh 1	0.7385	0.7375	0.7370	0.7364
Mesh 2	0.7624	0.7564	0.7559	0.7545
Mesh 3	0.7827	0.7818	0.7720	0.7611
Mesh 4	0.8153	0.8054	0.8001	0.7827



The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Projection Operators

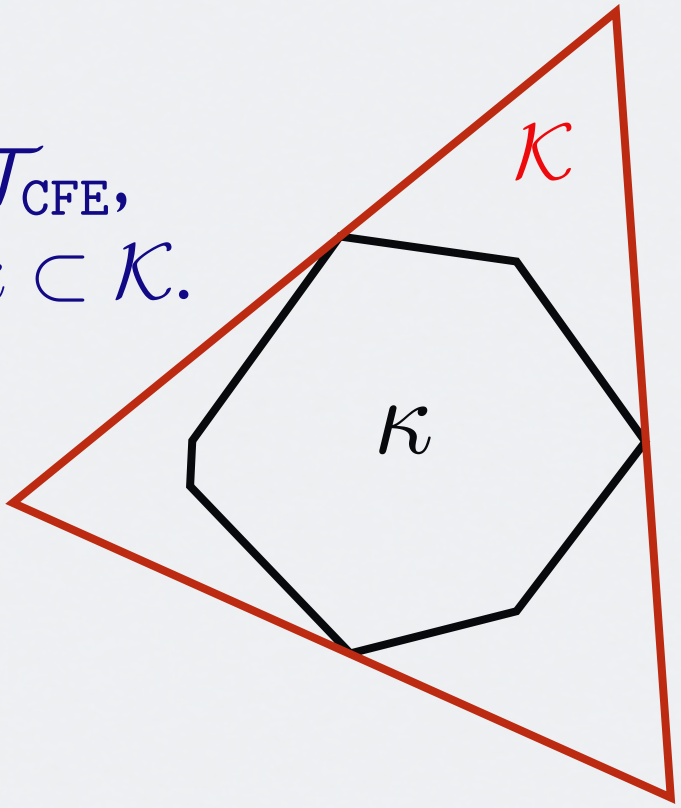
Let $\mathcal{T}_\sharp = \{\mathcal{K}\}$ denote a shape-regular covering of $\mathcal{T}_{\texttt{CFE}}$, such that for each $\kappa \in \mathcal{T}_{\texttt{CFE}}$, there exists $\mathcal{K} \in \mathcal{T}_\sharp$, $\kappa \subset \mathcal{K}$.



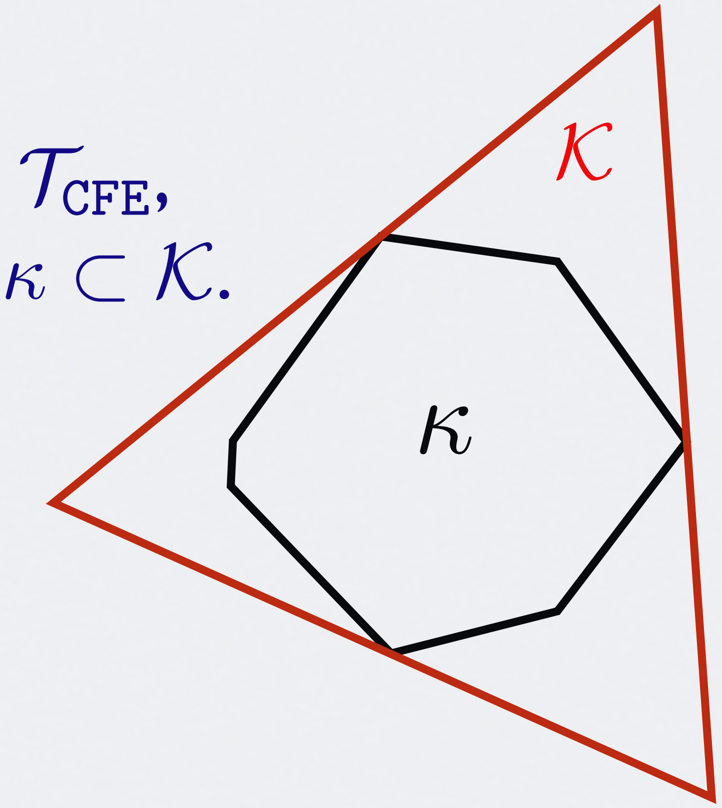
The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Projection Operators

Let $\mathcal{T}_\sharp = \{\mathcal{K}\}$ denote a shape-regular covering of $\mathcal{T}_{\texttt{CFE}}$, such that for each $\kappa \in \mathcal{T}_{\texttt{CFE}}$, there exists $\mathcal{K} \in \mathcal{T}_\sharp$, $\kappa \subset \mathcal{K}$.



Projection Operators



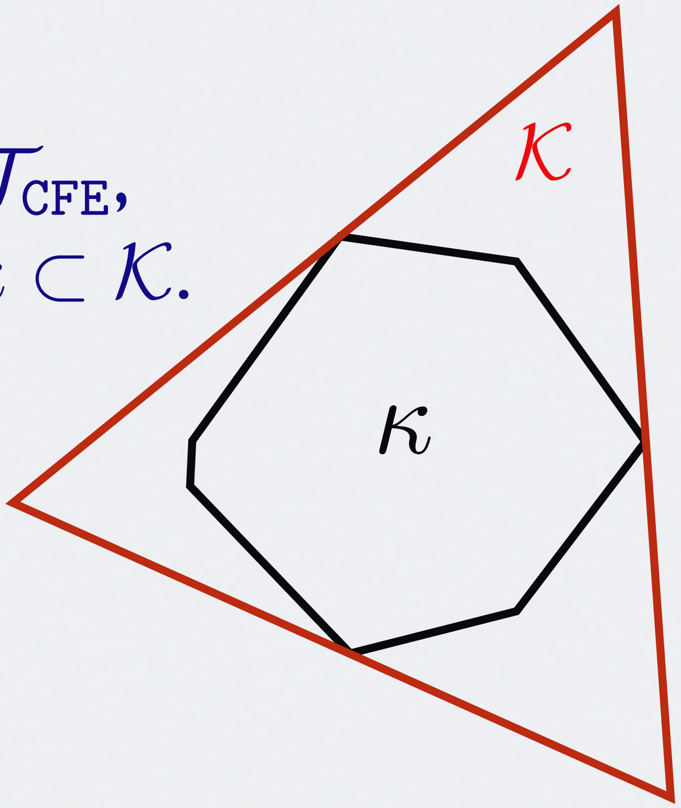
Let $\mathcal{T}_\sharp = \{\mathcal{K}\}$ denote a shape-regular covering of $\mathcal{T}_{\texttt{CFE}}$, such that for each $\kappa \in \mathcal{T}_{\texttt{CFE}}$, there exists $\mathcal{K} \in \mathcal{T}_\sharp$, $\kappa \subset \mathcal{K}$.

(A3) We assume that

$$\max_{\kappa \in \mathcal{T}_{\texttt{CFE}}} \text{card}\,\{\kappa' \in \mathcal{T}_{\texttt{CFE}} : \kappa' \cap \mathcal{K} \neq \emptyset,\ \mathcal{K} \in \mathcal{T}_\sharp\ \kappa \subset \mathcal{K}\} \leq \mathcal{O}_\Omega \quad \text{(uniformly)}$$

Projection Operators

Let $\mathcal{T}_\sharp = \{\mathcal{K}\}$ denote a shape-regular covering of $\mathcal{T}_{\texttt{CFE}}$, such that for each $\kappa \in \mathcal{T}_{\texttt{CFE}}$, there exists $\mathcal{K} \in \mathcal{T}_\sharp$, $\kappa \subset \mathcal{K}$.



(A3) We assume that

$$\max_{\kappa \in \mathcal{T}_{\texttt{CFE}}} \text{card}\left\{\kappa' \in \mathcal{T}_{\texttt{CFE}} : \kappa' \cap \mathcal{K} \neq \emptyset,\ \mathcal{K} \in \mathcal{T}_\sharp\ \kappa \subset \mathcal{K}\right\} \leq \mathcal{O}_\Omega \quad \text{(uniformly)}$$

We write $\tilde{\Pi}_p v = \Pi_p(\mathfrak{E}v|_{\mathcal{K}})|_\kappa$.

- Π_p: Projector on $\mathcal{K}$ (standard element shape).
- $\mathfrak{E}$: Extension operator.

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Theorem (Cangiani, Georgoulis, & H, 2013)

For $s_\kappa = \min\{p_\kappa + 1, k_\kappa\}$ and $p_\kappa \geq 1$, the following bound holds:

$$|||\, u - u_h \,|||^2_{\mathrm{DG}} \leq C \sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(k_\kappa - 1)}} \left(1 + \mathcal{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa)\right) \|\mathfrak{E}u\|^2_{H^{k_\kappa}(\mathcal{K})}.$$

Theorem (Cangiani, Georgoulis, & H, 2013)

For $s_\kappa = \min\{p_\kappa + 1, k_\kappa\}$ and $p_\kappa \geq 1$, the following bound holds:

$$|||\, u - u_h \,|||_{\mathrm{DG}}^2 \leq C \sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(k_\kappa - 1)}} \left(1 + \mathcal{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa)\right) \|\mathfrak{E}u\|_{H^{k_\kappa}(\mathcal{K})}^2 .$$

$$\mathcal{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa) = p_\kappa h_\kappa^{-d} \sum_{F \subset \partial\kappa} C_m(p_\kappa, \kappa, F)\sigma^{-1}|F|$$

$$+ p_\kappa^2 |\kappa|^{-1} \sum_{F \subset \partial\kappa} C_{\mathrm{INV}}(p_\kappa, \kappa, F)\sigma^{-1}|F| + h_\kappa^{-d+2} p_\kappa^{-1} \sum_{F \subset \partial\kappa} C_m(p_\kappa, \kappa, F)\sigma|F|,$$

$$C_{\mathrm{INV}}(p, \kappa, F) := C_{\mathrm{inv}} \min\left\{ \frac{|\kappa|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, p^{2d} \right\},$$

$$C_m(p_\kappa, \kappa, F) = \min\left\{ \frac{h_\kappa^d}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, \frac{1}{p_\kappa^{1-d}} \right\}.$$

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Theorem (Cangiani, Georgoulis, & H, 2013)

For $s_\kappa = \min\{p_\kappa + 1, k_\kappa\}$ and $p_\kappa \geq 1$, the following bound holds:

$$|||\, u - u_h \,|||_{\mathrm{DG}}^2 \leq C \sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(k_\kappa - 1)}} \left(1 + \mathcal{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa)\right) \|\mathfrak{E}u\|_{H^{k_\kappa}(\mathcal{K})}^2 .$$

For uniform orders $p_\kappa = p \geq 1$, $h = \max_{\kappa \in \mathcal{T}} h_\kappa$, $s_\kappa = s$, $s = \min\{p+1, k\}$, $k > 1 + d/2$, and $\mathrm{diam}(F) \sim h_\kappa$, $F \subset \partial\kappa$, $\kappa \in \mathcal{T}_{\mathrm{CFE}}$, we get the bound

$$|||\, u - u_h \,|||_{\mathrm{DG}} \leq C \frac{h^{s-1}}{p^{k-3/2}} \|u\|_{H^k(\Omega)} .$$

cf. H., Schwab & Süli 2002.

Theorem (Cangiani, Dong, Georgoulis, & H, 2015)

For uniform orders we have that

$$|||\, u - u_h \,|||_{\mathrm{Hyp}} \leq C \, \frac{h^{s-1/2}}{p^{k-1}} \|u\|_{H^k(\Omega)}.$$

for $s = \min\{p+1, k\}$, $k > 1 + d/2$.

Proof

The proof is based on employing an inf-sup condition with respect to a stronger streamline-diffusion DGFEM norm.

Agglomeration-based Adaptivity

Error Estimation

- Energy norm based error estimation:

  Giani & H. 2014: Overlapping refined meshes, cf. Hackbusch & Sauter 1997

- Goal-oriented error estimation:

$$J(u) - J(u_h) = \sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \eta_\kappa ,$$

where $\eta_\kappa = \eta_\kappa(u_h, z - z_h)$ and z is the adjoint/dual solution.

Error Estimation

- Energy norm based error estimation:

  Giani & H. 2014: Overlapping refined meshes, cf. Hackbusch & Sauter 1997

- Goal-oriented error estimation:

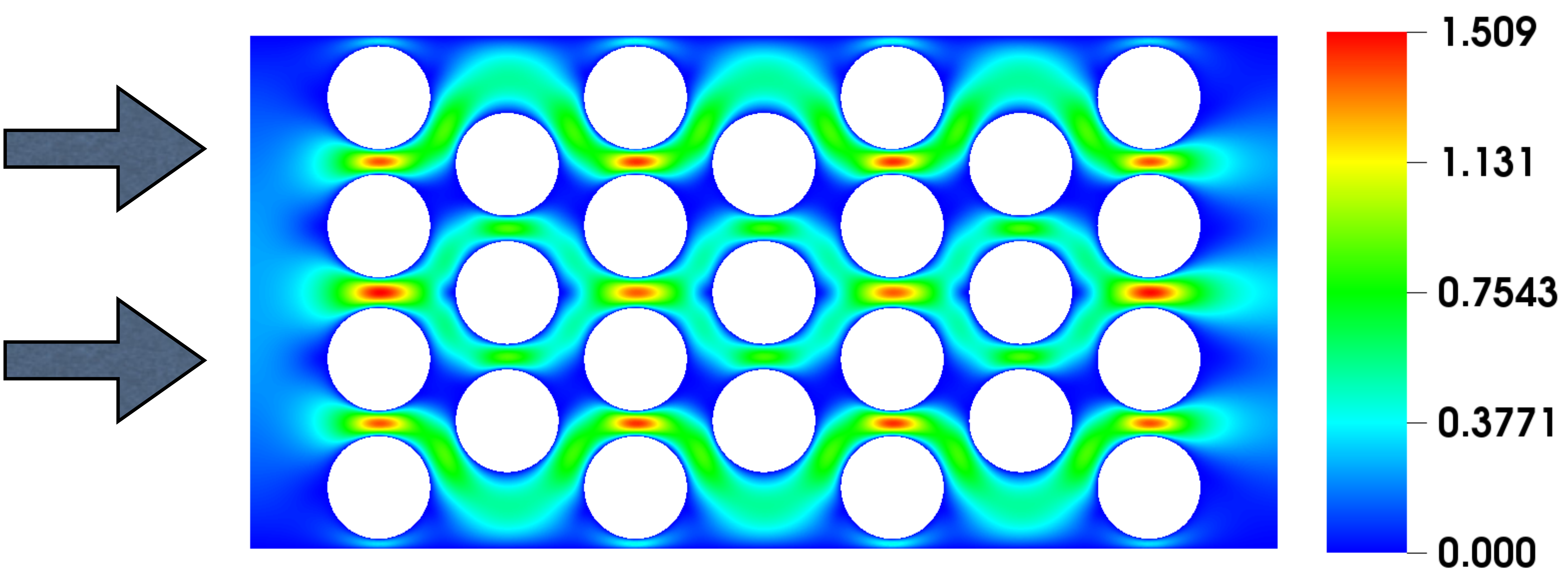
$$J(u) - J(u_h) = \sum_{\kappa \in \mathcal{T}_{\texttt{CFE}}} \eta_\kappa,$$

where $\eta_\kappa = \eta_\kappa(u_h, z - z_h)$ and z is the adjoint/dual solution.

Adaptivity

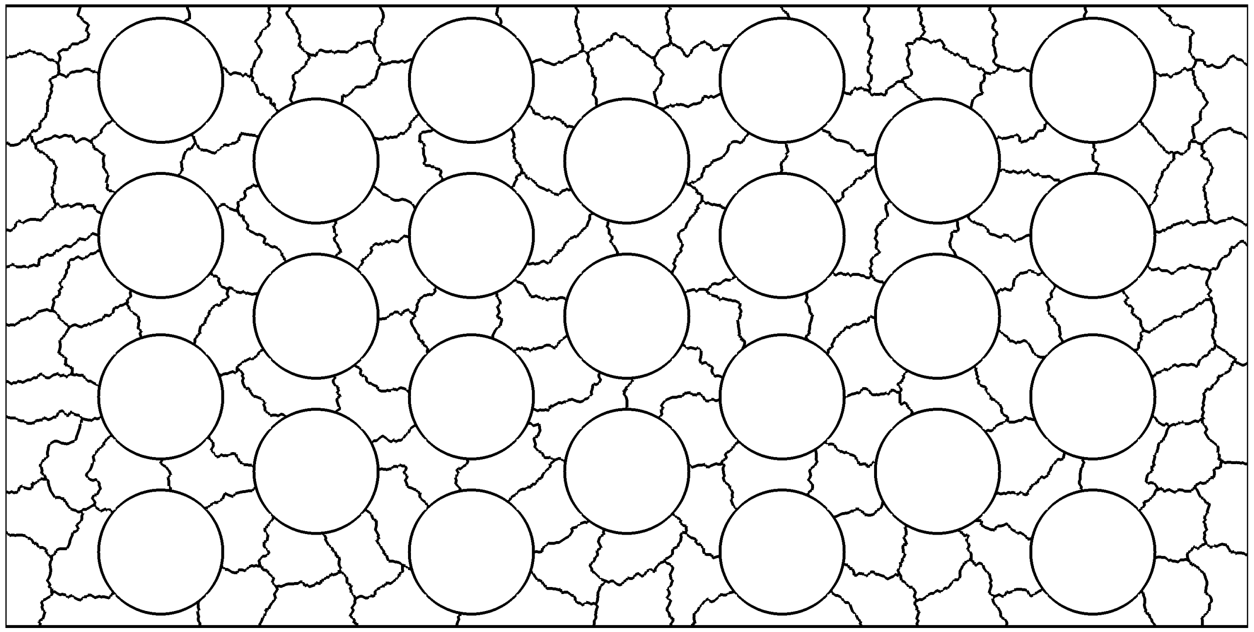
- Input fine geometry-conforming (standard) mesh $\mathcal{T}_{\texttt{fine}}$.
- Agglomerate $\mathcal{T}_{\texttt{fine}}$ into a user defined number of partitions ($\mathcal{T}_{\texttt{CFE}}$).
- Adaptively refine $\kappa \in \mathcal{T}_{\texttt{CFE}}$ using agglomeration based on $|\eta_\kappa|$.
- Elements in $\mathcal{T}_{\texttt{fine}}$ only get refined if further resolution is required.

$Re = 10$: DWR Refinement, with $J(\mathbf{u}, p) = p(1.9, 0.3) \approx 1.74825 \times 10^{-2}$

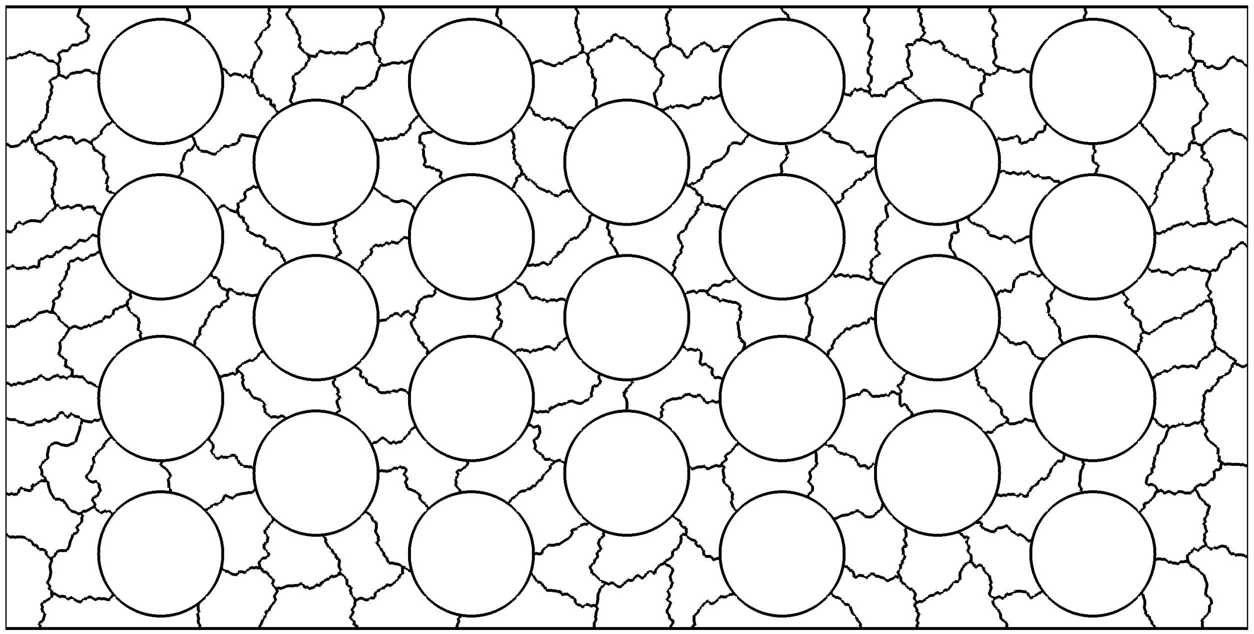


Rejniak, Estrella, Chen, Cohen, Lloyd, & Morse 2013

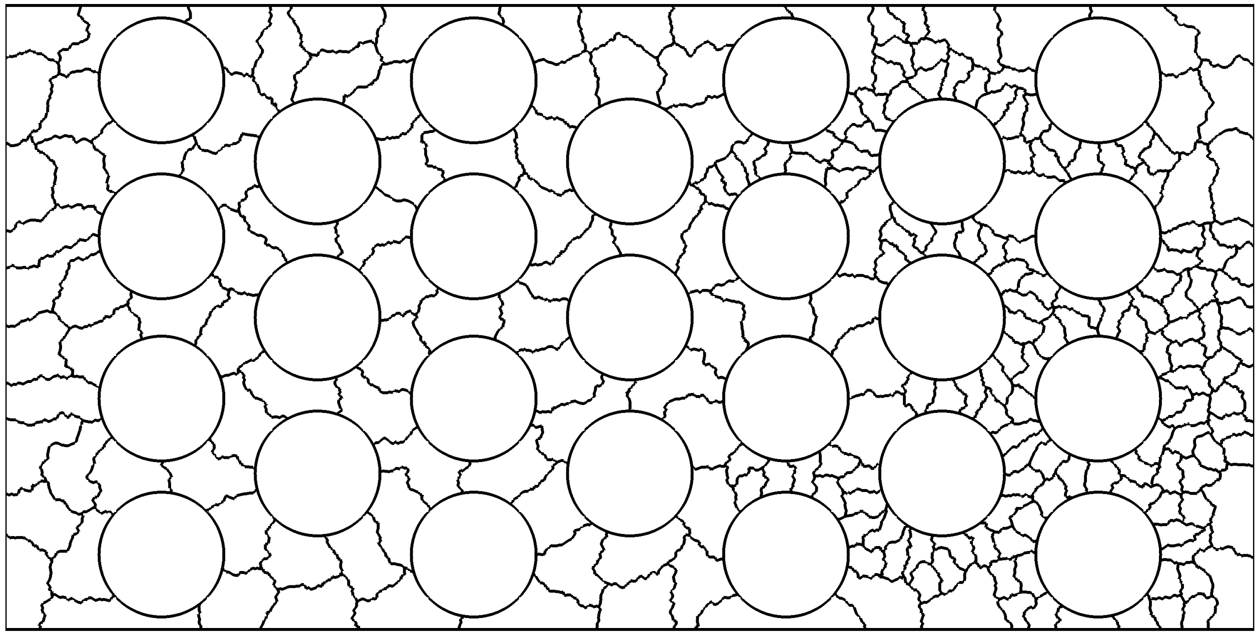
Mesh 1: 128 Elements



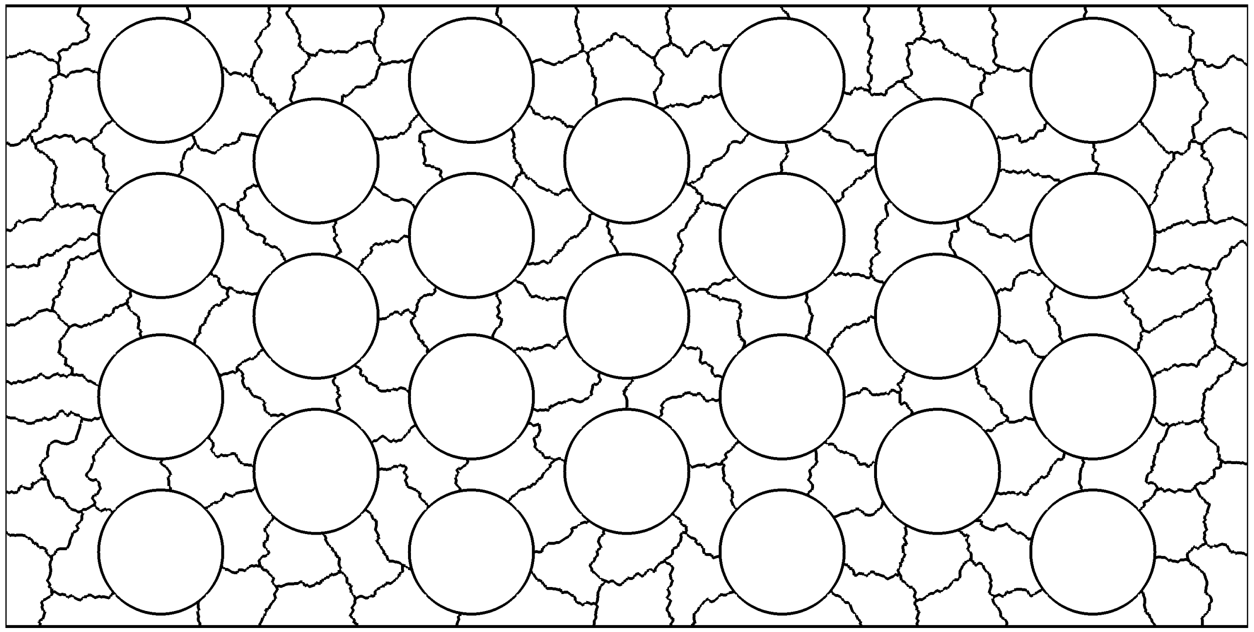
Mesh 1: 128 Elements



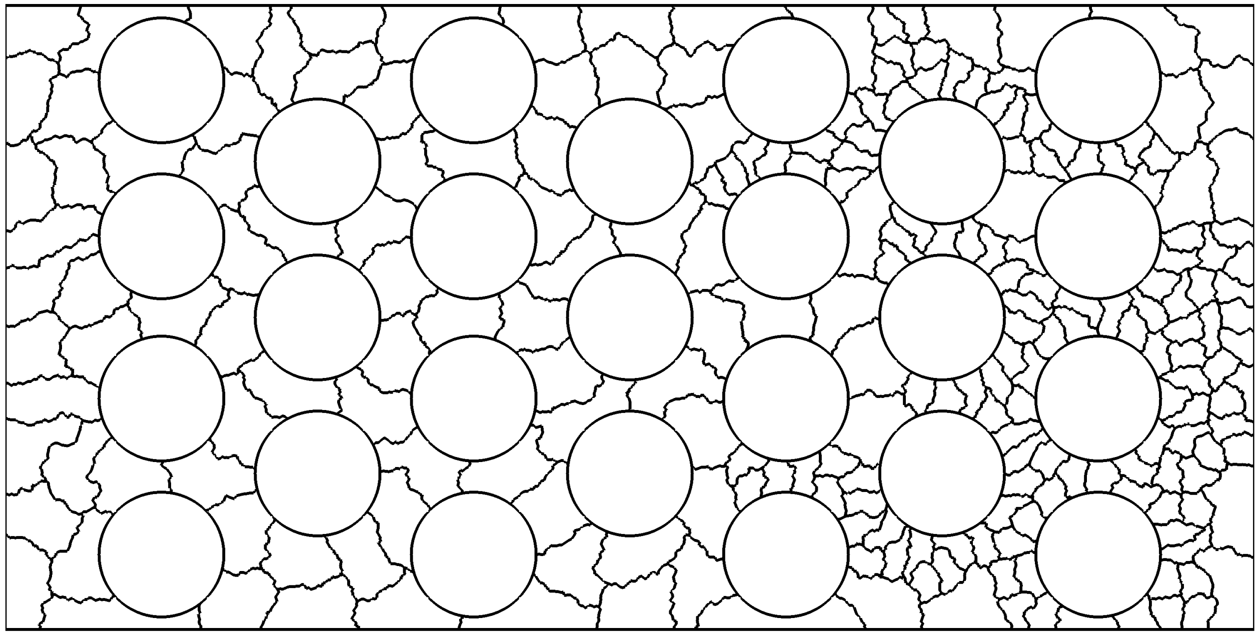
Mesh 2: 224 Elements



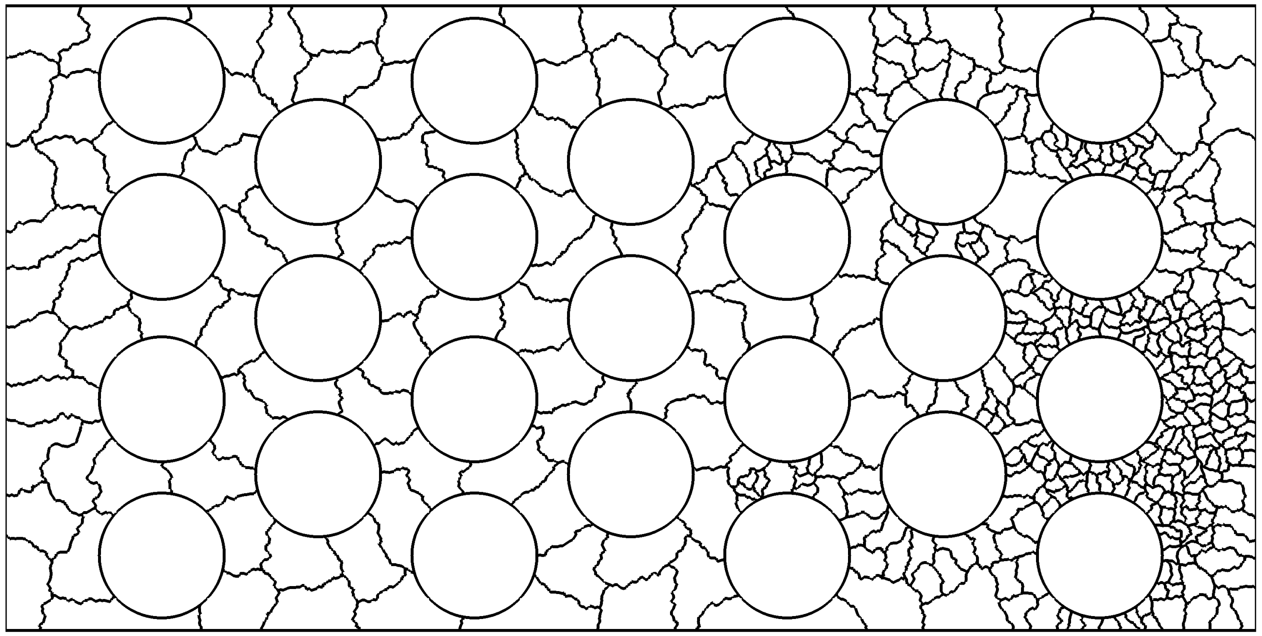
Mesh 1: 128 Elements



Mesh 2: 224 Elements

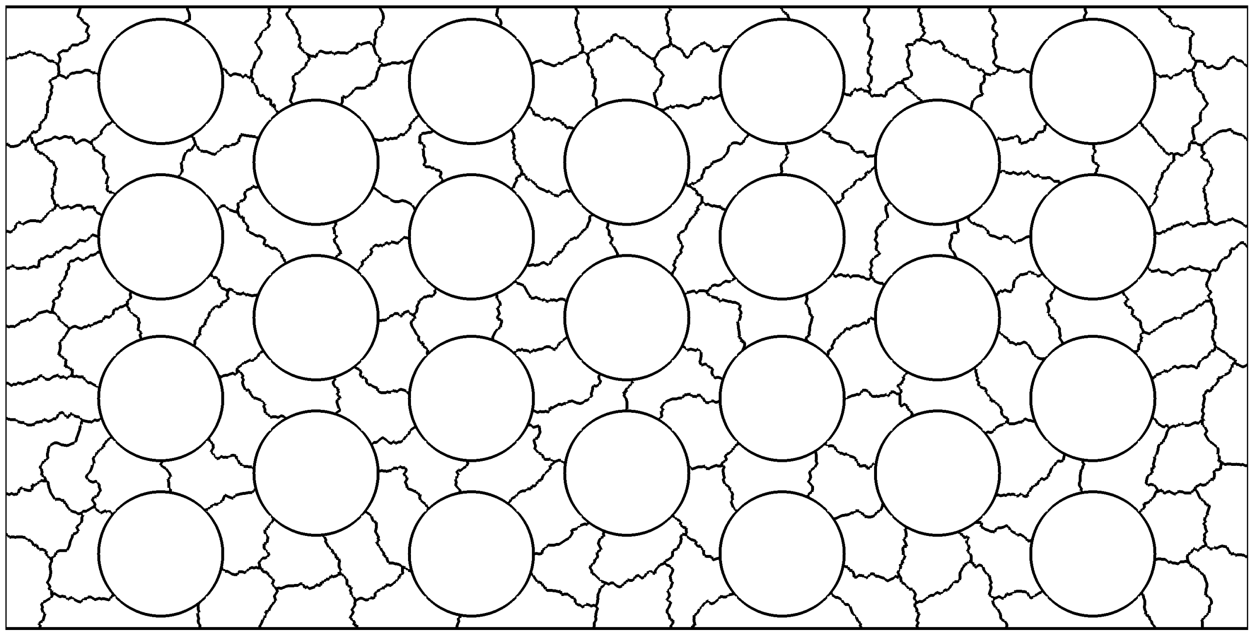


Mesh 3: 392 Elements

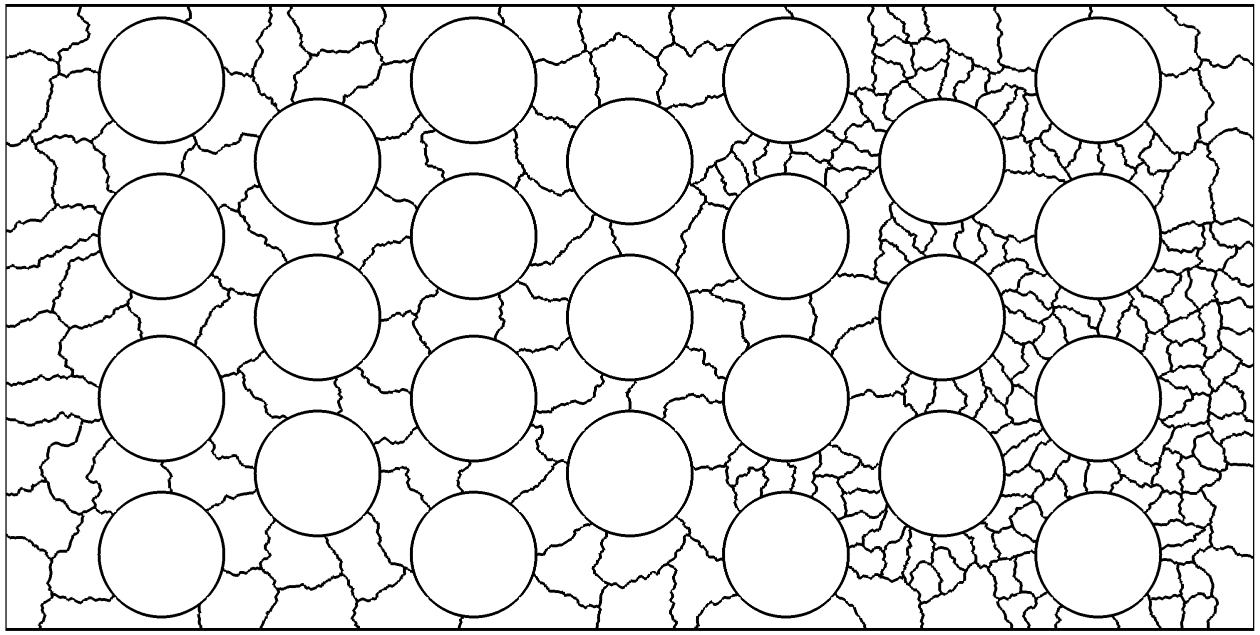


Interstitial Fluid Modelling

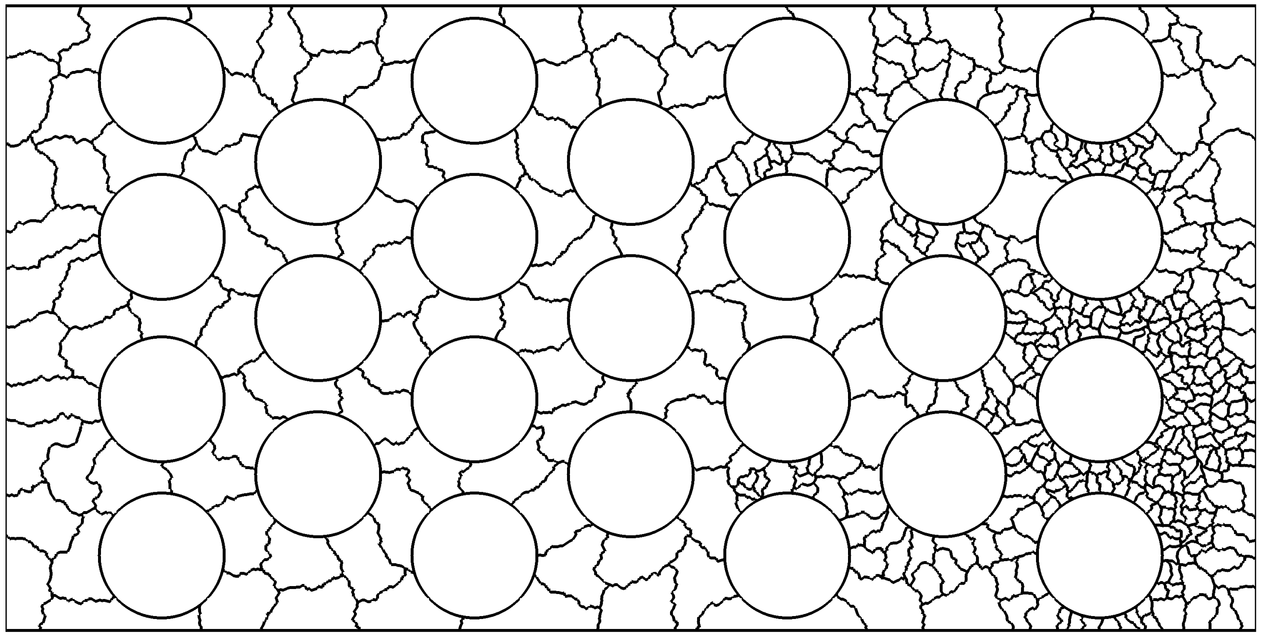
Mesh 1: 128 Elements



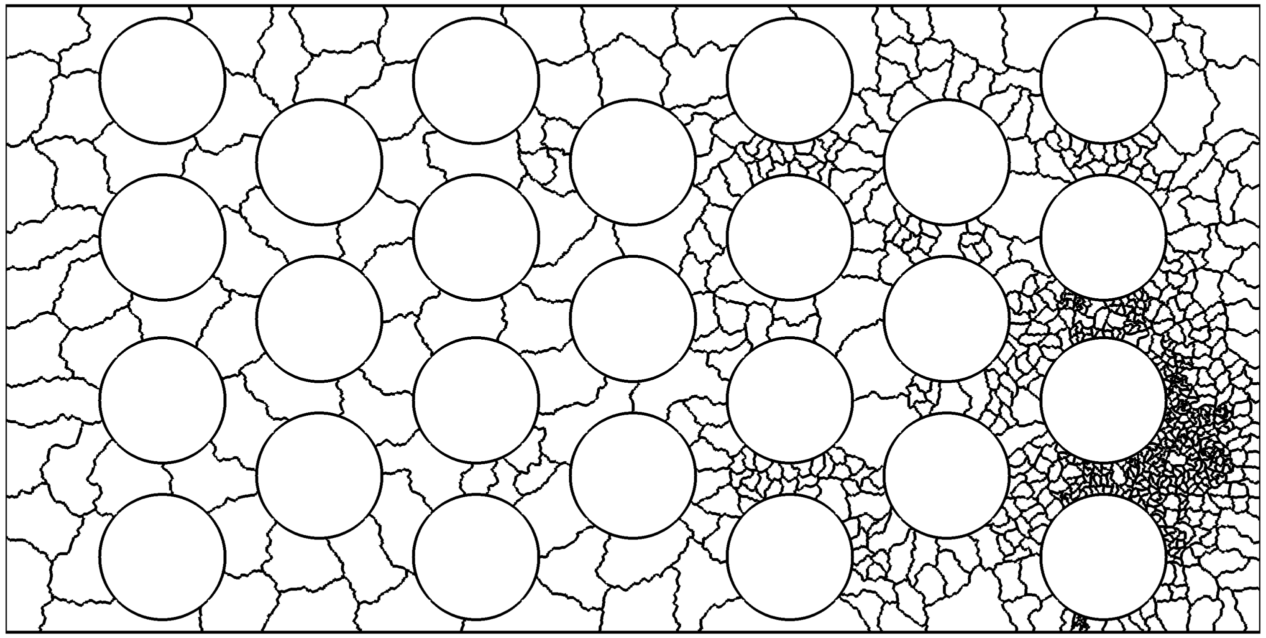
Mesh 2: 224 Elements



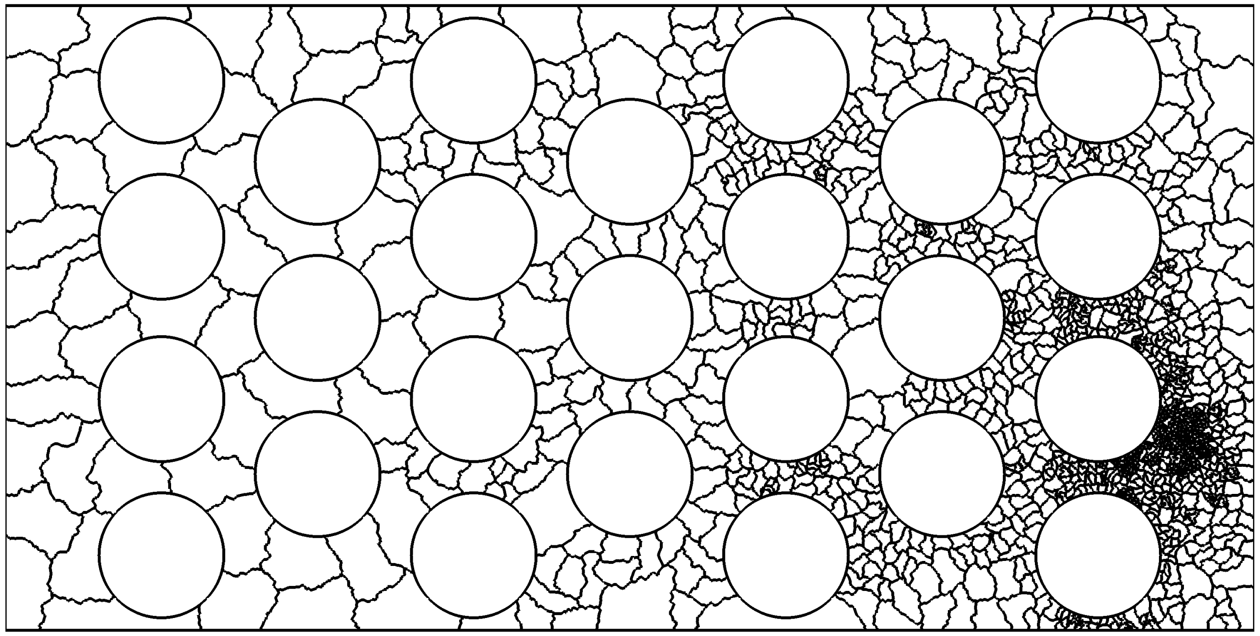
Mesh 3: 392 Elements



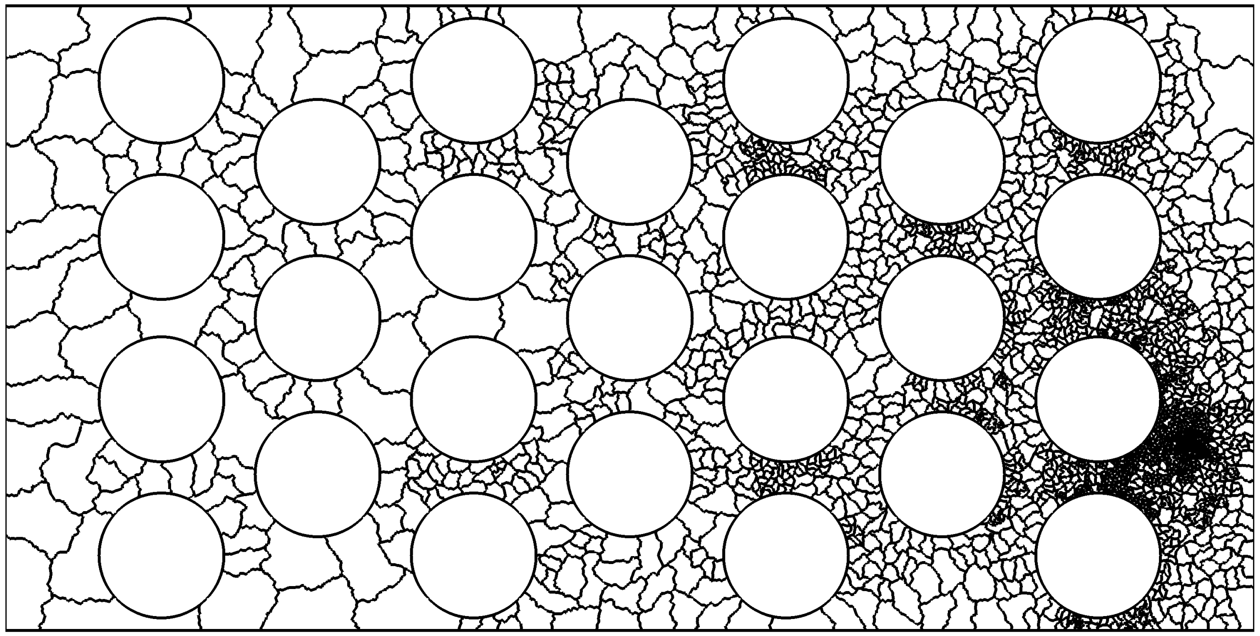
Mesh 4: 686 Elements



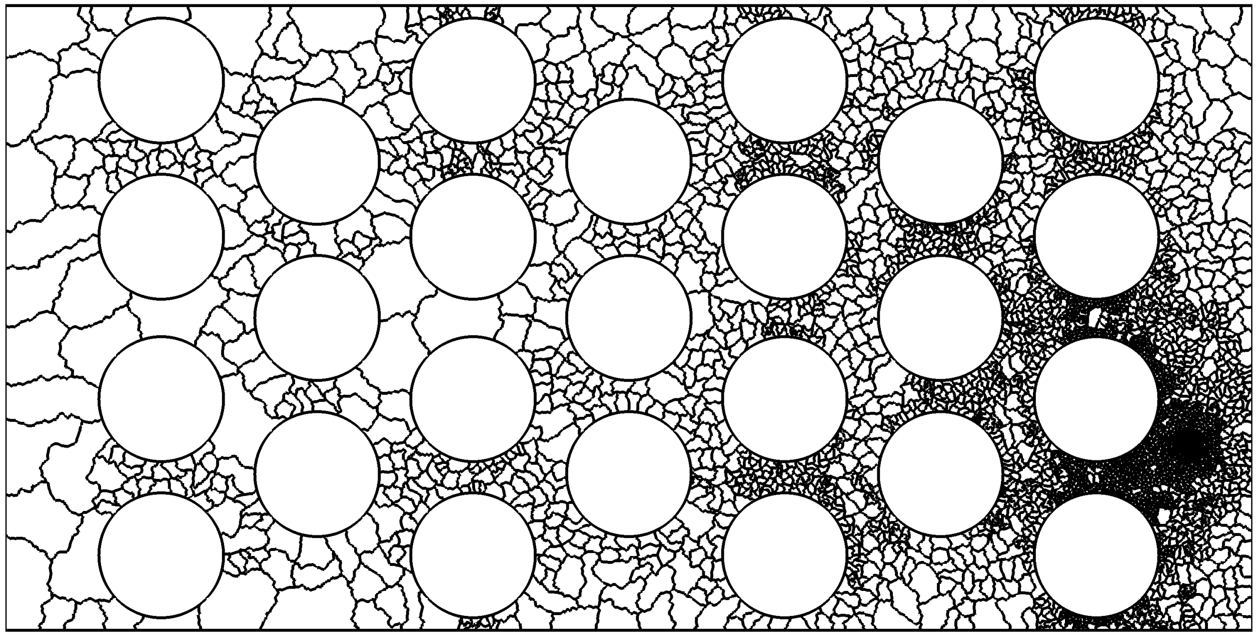
Mesh 5: 1199 Elements



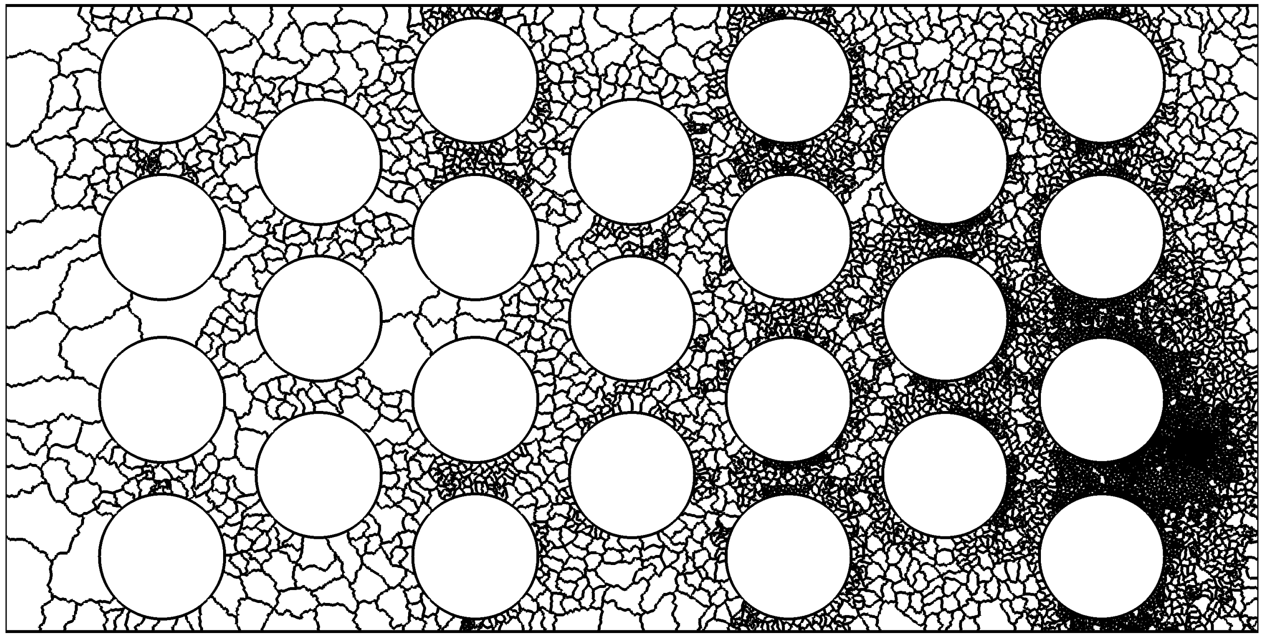
Mesh 6: 1994 Elements



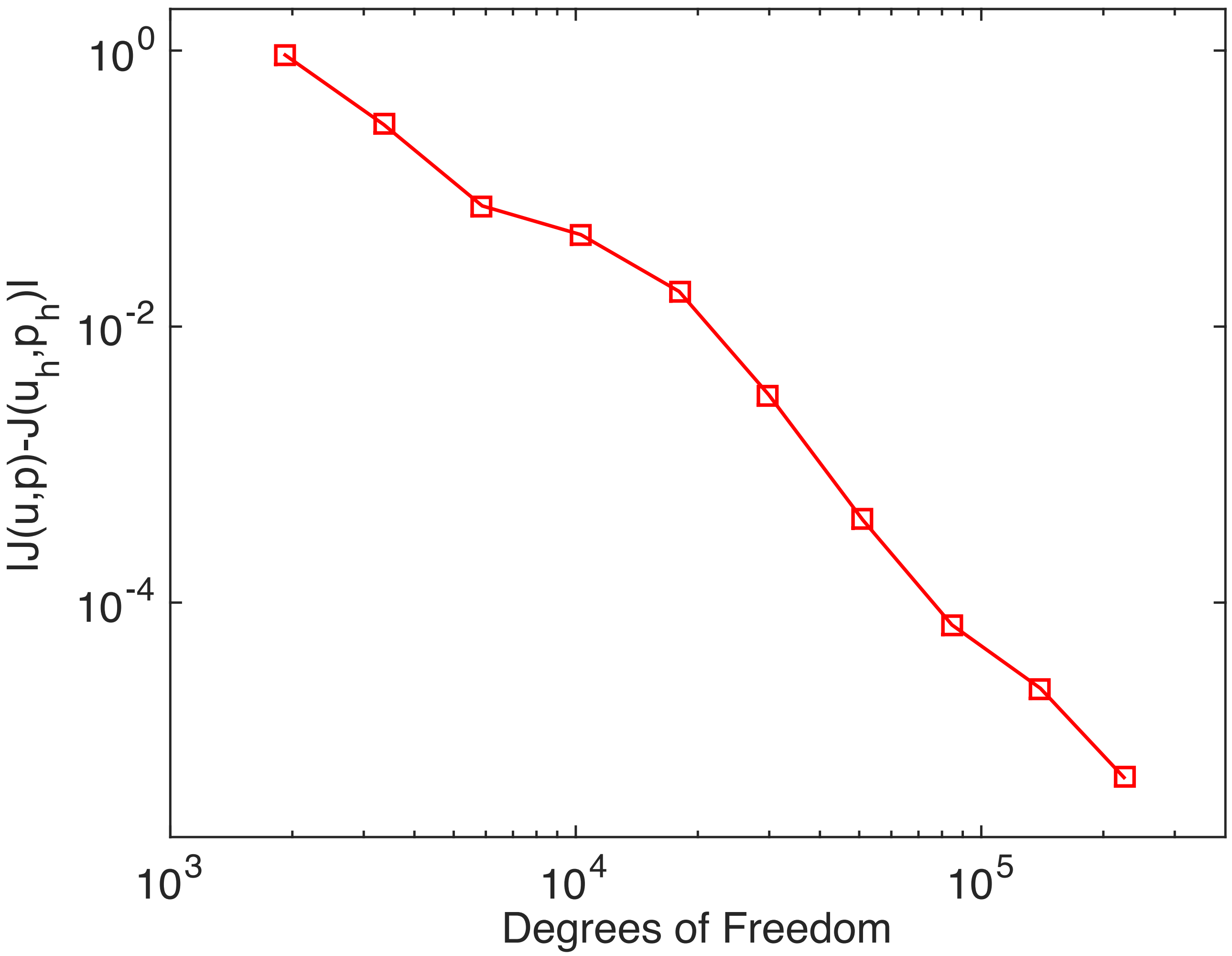
Mesh 7: 3396 Elements



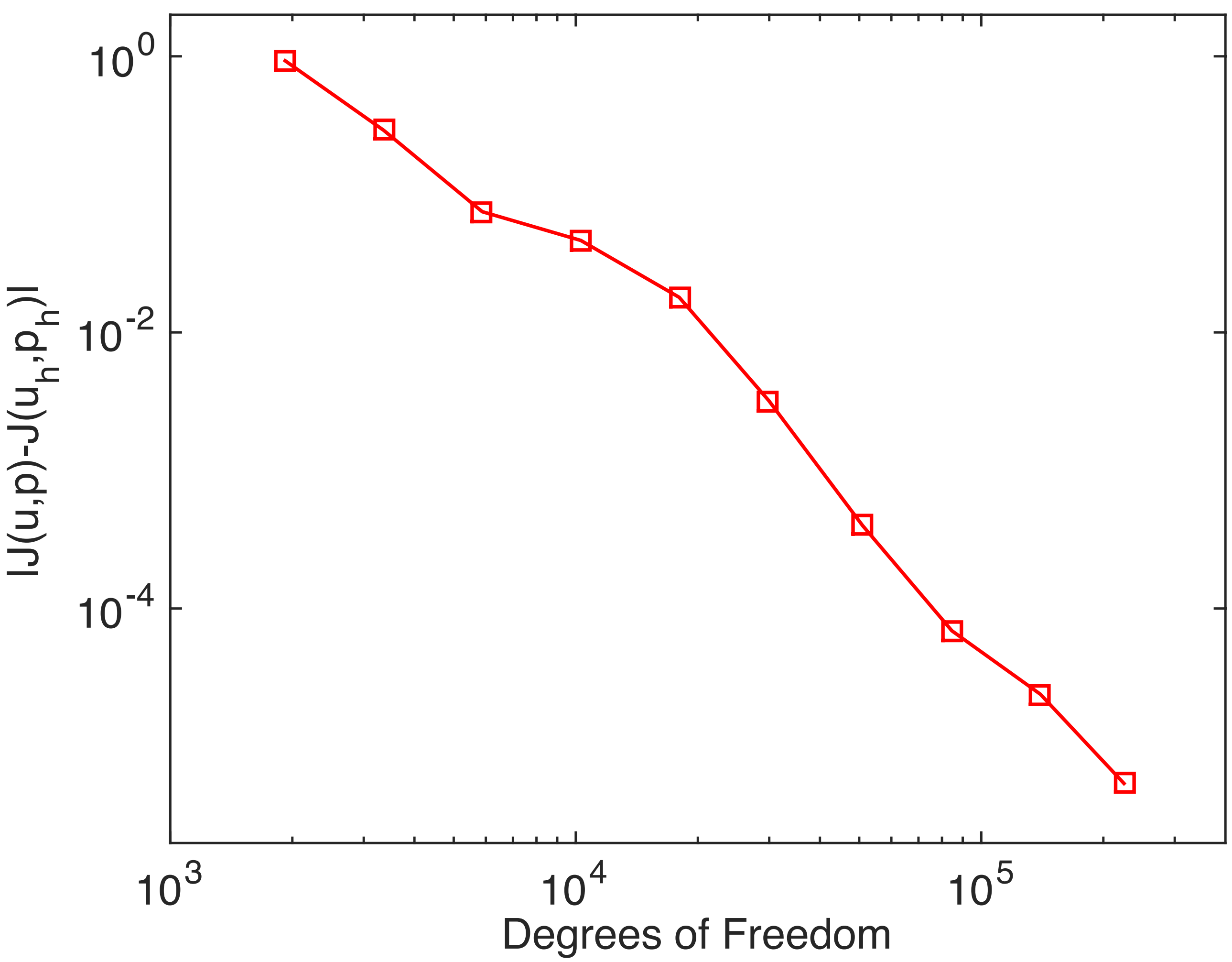
Mesh 8: 5642 Elements

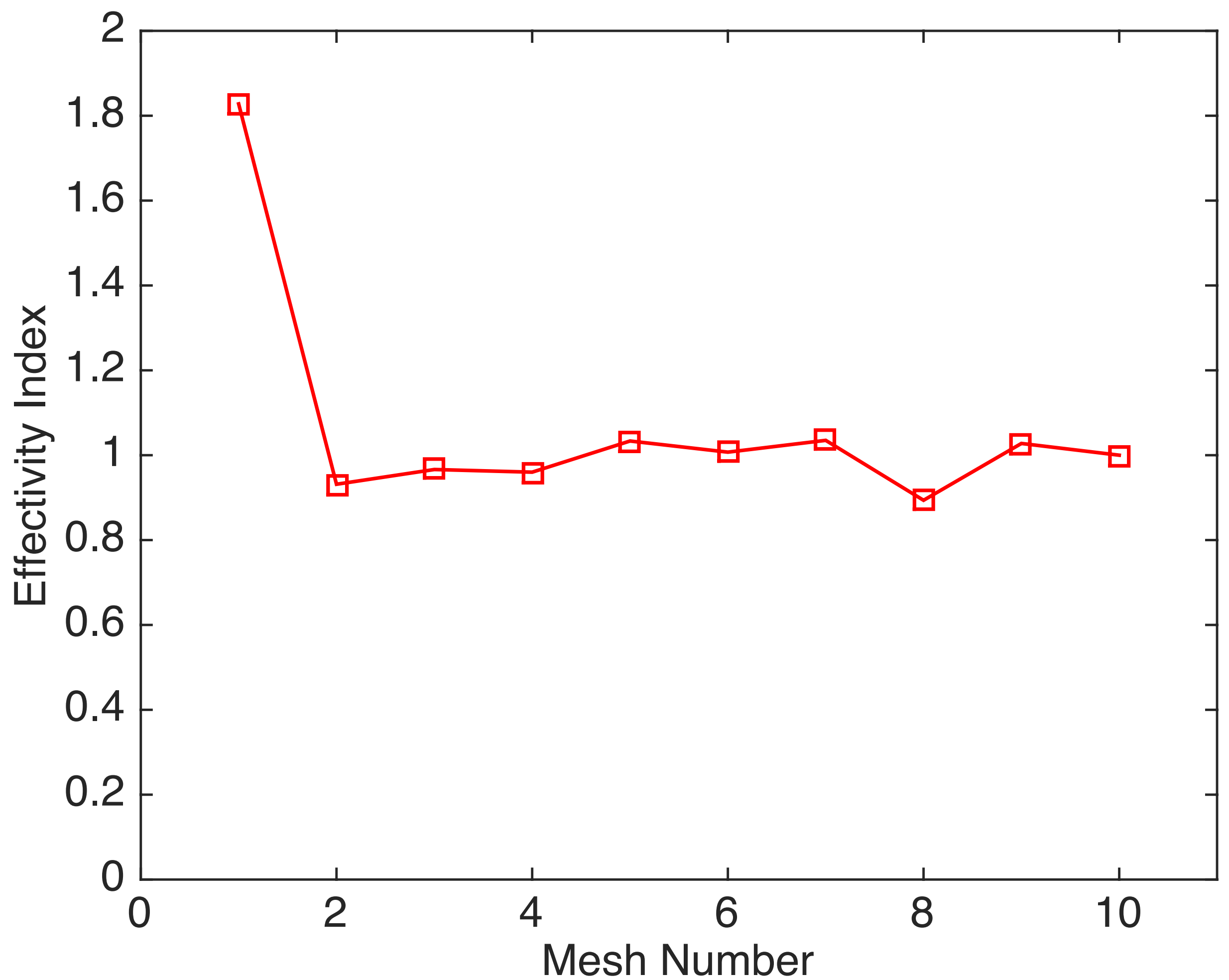


$Re = 10$: DWR Refinement, with $J(\mathbf{u}, p) = p(1.9, 0.3) \approx 1.74825 \times 10^{-2}$



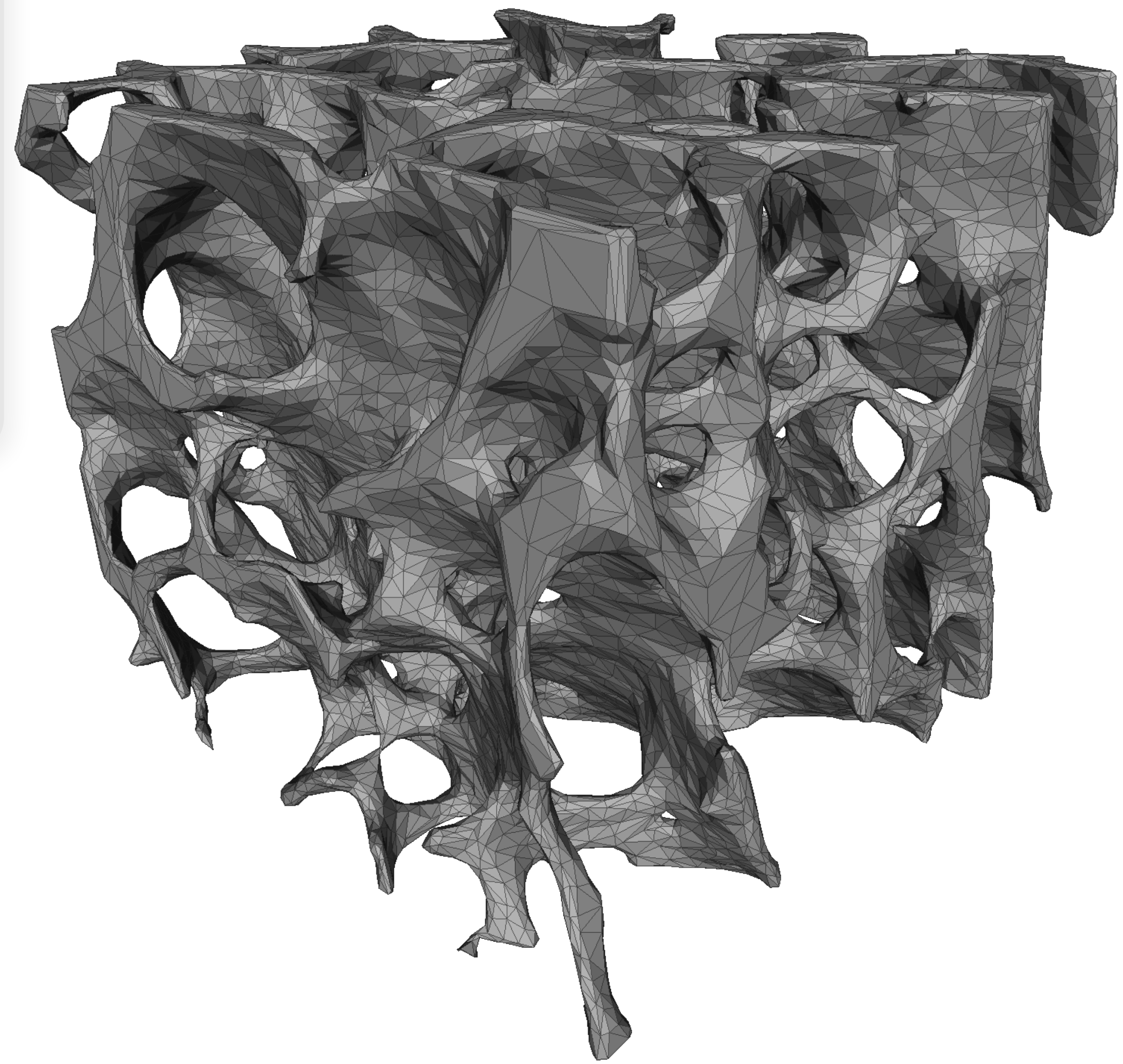
$Re = 10$: DWR Refinement, with $J(\mathbf{u}, p) = p(1.9, 0.3) \approx 1.74825 \times 10^{-2}$





Modelling Trabecular Bone

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) &= 0 \quad \text{in } \Omega, \\
\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} &= 0 \quad \text{on } \partial\Omega_{\text{int}}, \\
\mathbf{u} \cdot \mathbf{n} &= g_n \quad \text{on } \partial\Omega_{\text{box}}, \\
\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \cdot \mathbf{t} &= 0 \quad \text{on } \partial\Omega_{\text{box}}.
\end{aligned}
$$



Verhoosel, van Zwieten, van Rietbergen & de Borst 2015

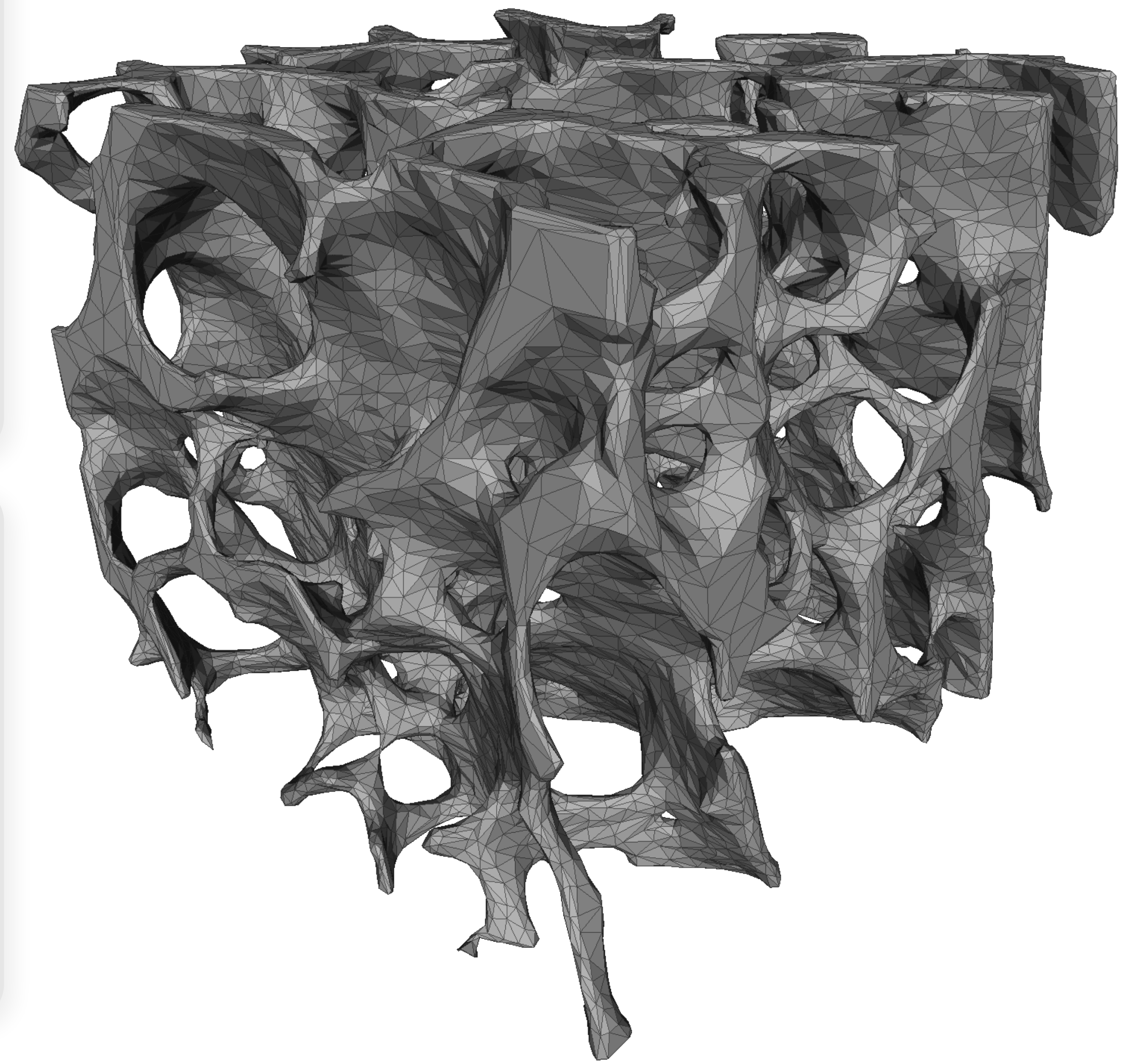
Modelling Trabecular Bone

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) &= 0 \quad \text{in } \Omega, \\
\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} &= 0 \quad \text{on } \partial\Omega_{\text{int}}, \\
\mathbf{u} \cdot \mathbf{n} &= g_n \quad \text{on } \partial\Omega_{\text{box}}, \\
\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \cdot \mathbf{t} &= 0 \quad \text{on } \partial\Omega_{\text{box}}.
\end{aligned}
$$

$$
J(\mathbf{u}) = \frac{1}{E} \frac{1}{g_n^{\text{top}}} \frac{h_{\text{box}}}{|\Omega_{\text{box}}|} \int_\Omega \boldsymbol{\sigma}_{33} d\mathbf{x},
$$

$$
g_n^{\text{top}} = 0.01 h_{\text{box}}.
$$

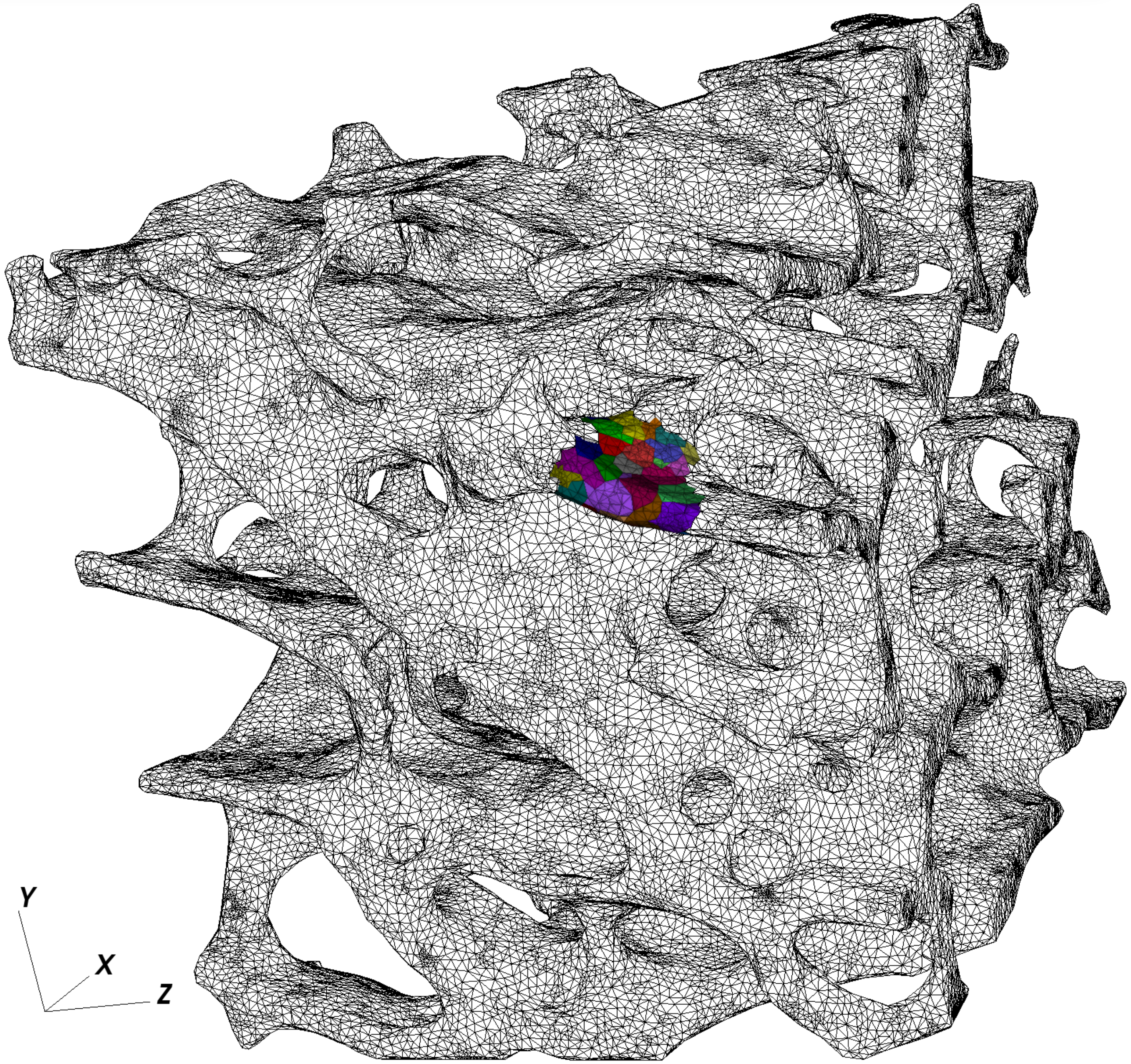
$E = 10\text{GPa}$ and $\nu = 0.3$.



Verhoosel, van Zwieten, van Rietbergen & de Borst 2015

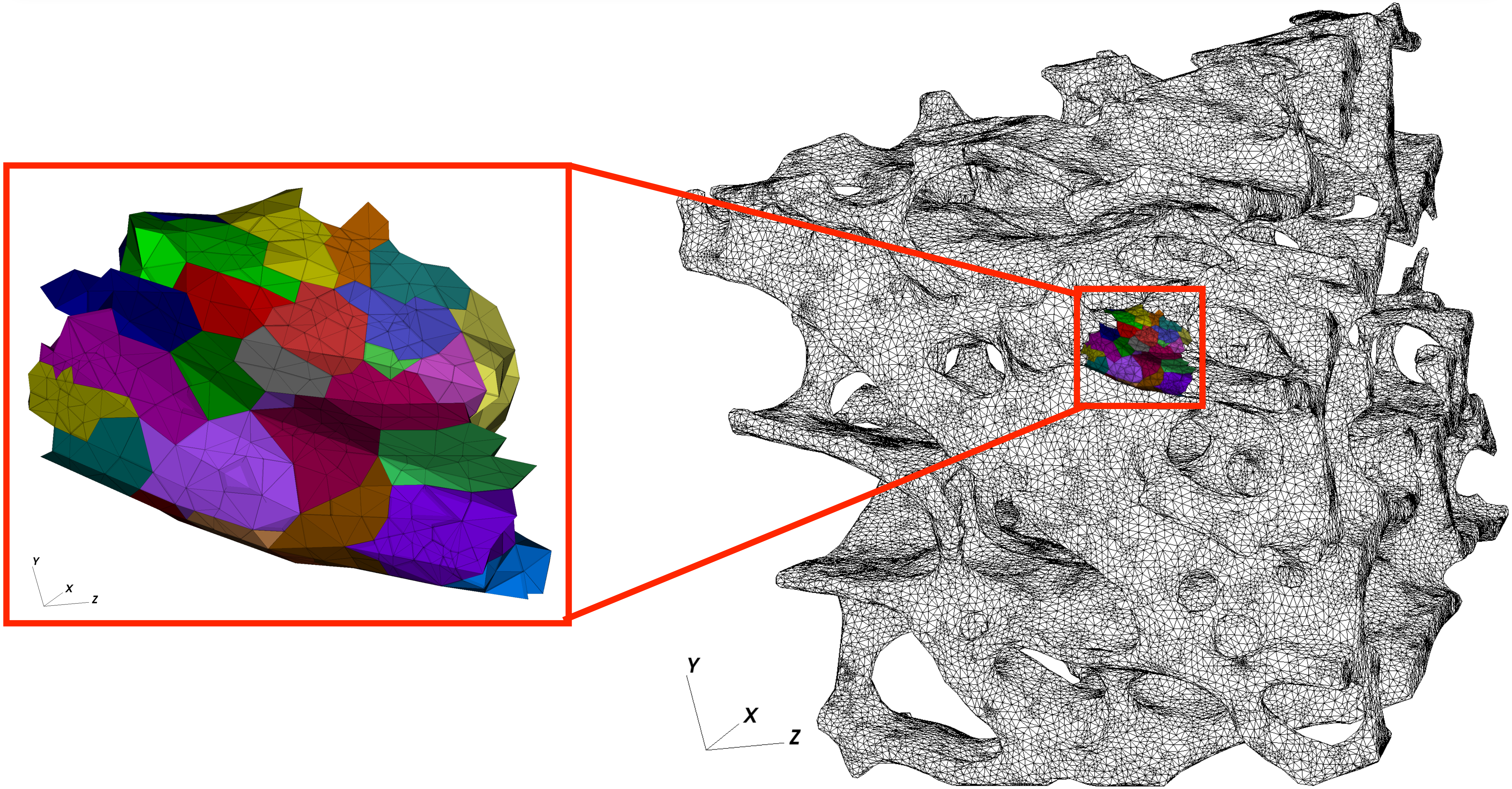
Modelling Trabecular Bone

Fine mesh consists of 1.2M elements; Agglomerated mesh with 8K elements.



Verhoosel, van Zwieten, van Rietbergen & de Borst 2015

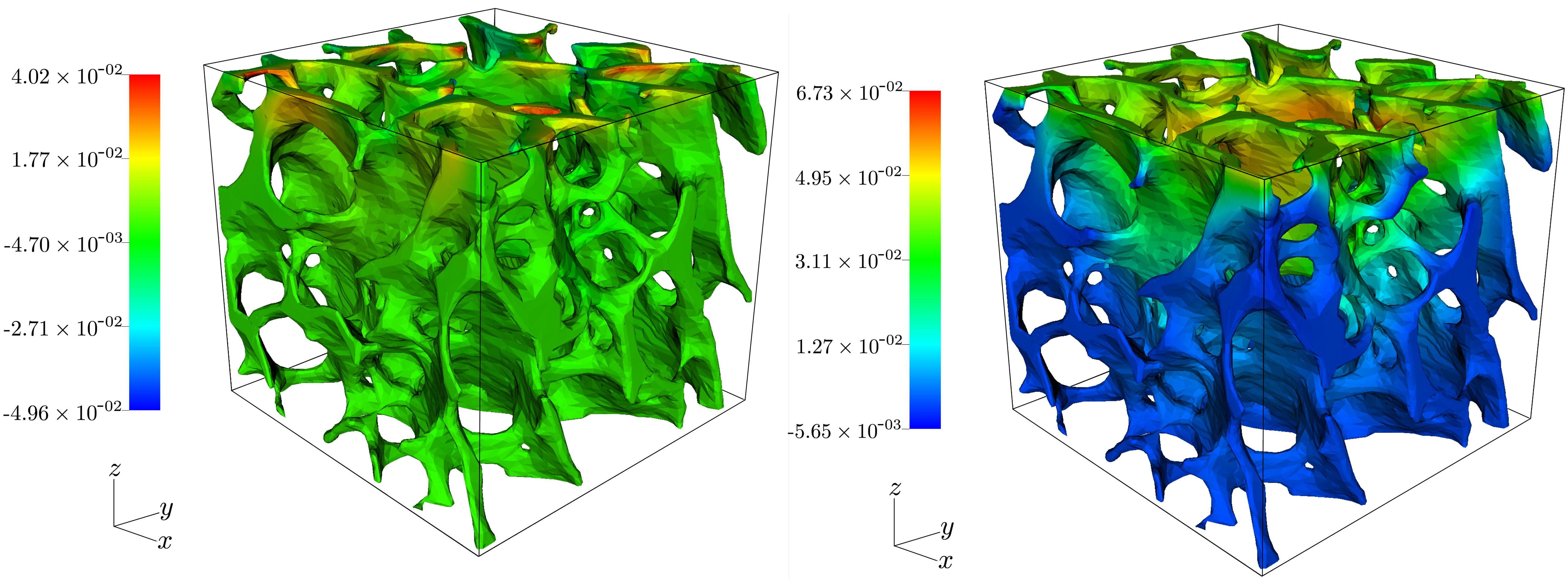
Fine mesh consists of 1.2M elements; Agglomerated mesh with 8K elements.



Verhoosel, van Zwieten, van Rietbergen & de Borst 2015

Modelling Trabecular Bone

Fine mesh consists of 1.2M elements; Agglomerated mesh with 8K elements.

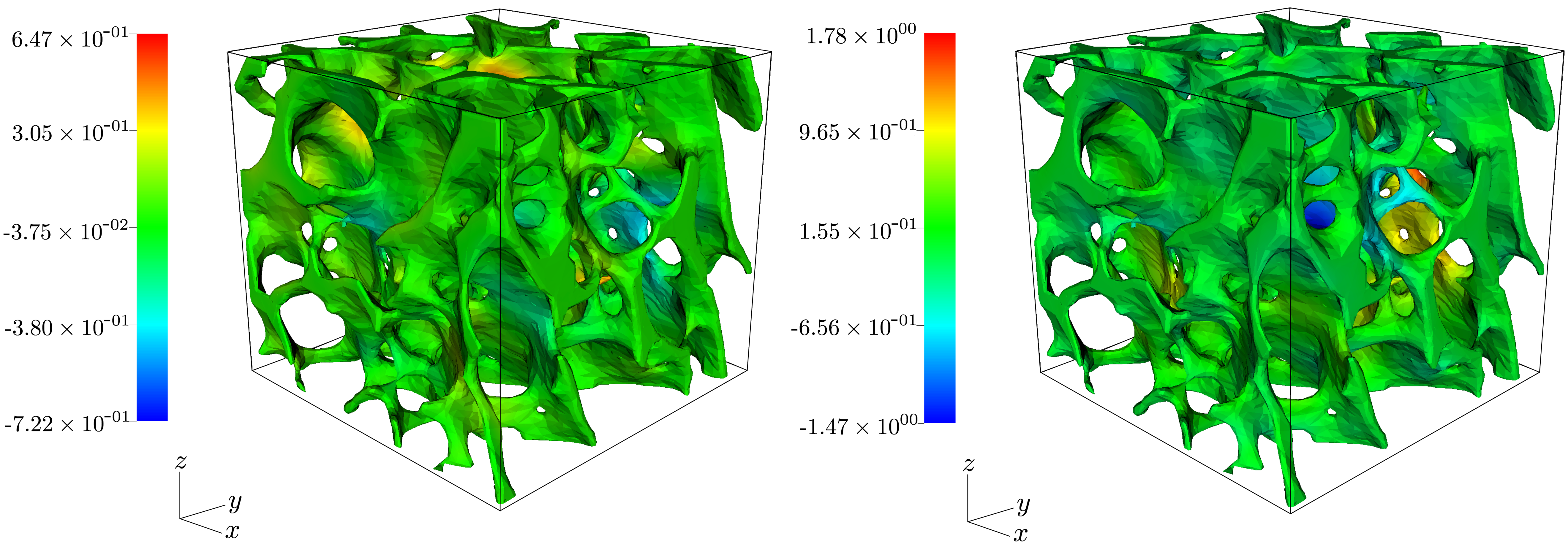


u_1 u_3

Verhoosel, van Zwieten, van Rietbergen & de Borst 2015

Modelling Trabecular Bone

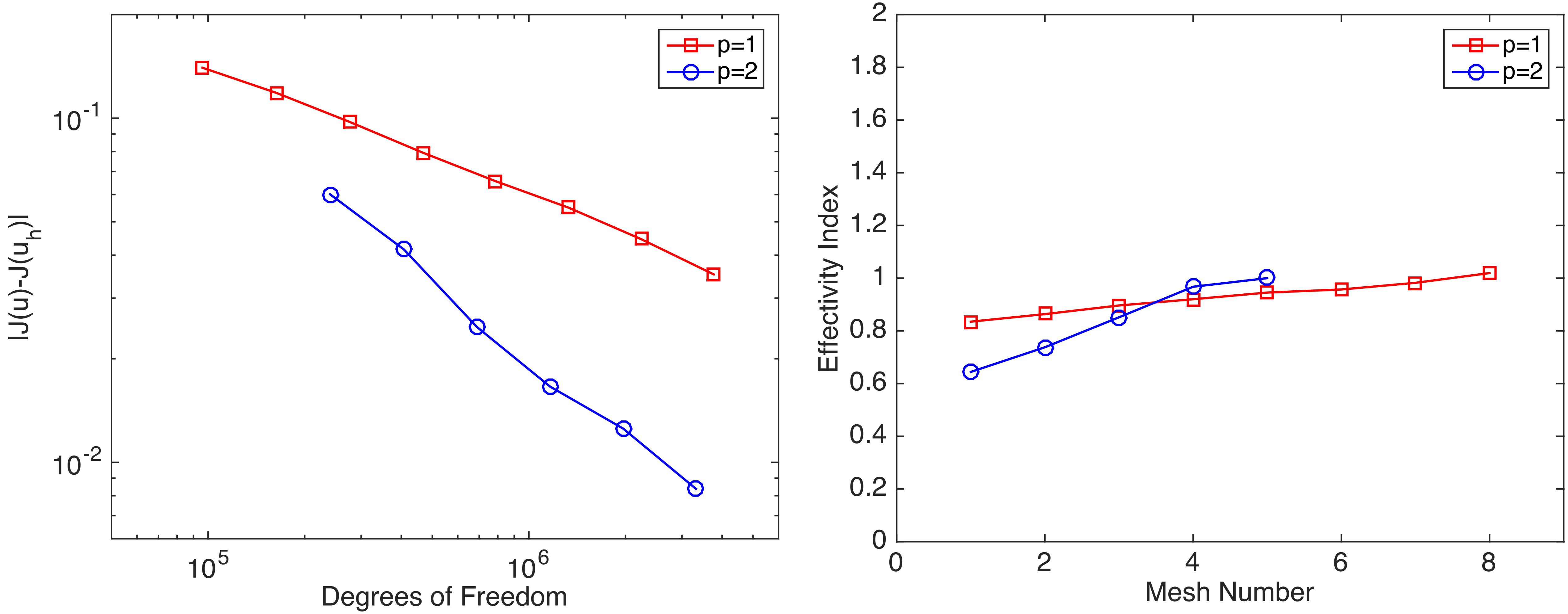
Fine mesh consists of 1.2M elements; Agglomerated mesh with 8K elements.



z_1 z_3

Verhoosel, van Zwieten, van Rietbergen & de Borst 2015

Modelling Trabecular Bone



Verhoosel, van Zwieten, van Rietbergen & de Borst 2015

Domain Decomposition Preconditioners

Domain Decomposition Preconditioning

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Goal

$$hp\text{–DGFEM} \longrightarrow Au = f$$

A is a large sparse, s.p.d. and ill-conditioned $\kappa(A) = \mathcal{O}(p^4 h^{-2})$

Goal

$$hp\text{–DGFEM} \longrightarrow Au = f$$

A is a large sparse, s.p.d. and ill-conditioned $\kappa(A) = \mathcal{O}(p^4 h^{-2})$

- Efficiently solve the algebraic linear system arising from the *hp*-DGFEM.
- Solver should be effective for both *h*- and *p*-version.

Domain Decomposition Preconditioning

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Goal

$$hp\text{–DGFEM} \longrightarrow Au = f$$

A is a large sparse, s.p.d. and ill-conditioned $\kappa(A) = \mathcal{O}(p^4 h^{-2})$

- Efficiently solve the algebraic linear system arising from the hp-DGFEM.
- Solver should be effective for both *h*- and *p*-version.

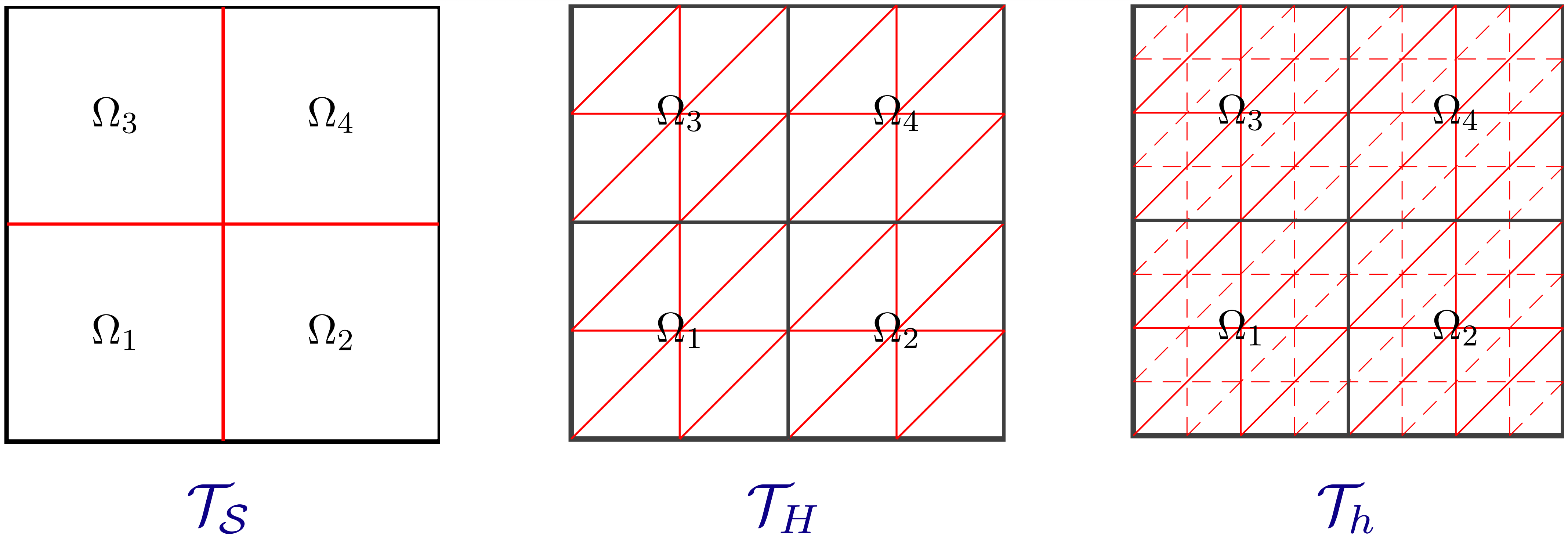
Domain Decomposition

$\Rightarrow$ Solve the PDE on $\Omega = \cup_{i=1}^{N} \Omega_i$.

$\Rightarrow$ Solve a series of local problems on each subdomain $\Omega_i, i = 1, \ldots, N$.

- Divide and Conquer: capability to treat large-scale problems.
- Parallelization: Local problems can be run on different processors.

Notation



- $\mathcal{T}_{\mathcal{S}} = \{\Omega_i\}_{i=1}^N$: Non-overlapping subdomain partition.
- $\mathcal{T}_h$: Fine mesh.
- $\mathcal{T}_H \equiv \mathcal{T}_{\texttt{CFE}}$: Coarse (agglomerated) mesh.

Assumption

$$\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_H \subseteq \mathcal{T}_h$$

Coarse Solver (DGFEM)

$$B_{\mathtt{DG}_0}(u_0, v_0) := B_{\mathtt{CDG}}(u_0, v_0) \qquad \forall u_0, v_0 \in V(\mathcal{T}_H, q).$$

Schwarz Preconditioners for *hp*-DGFEM

Coarse Solver (DGFEM)

$$B_{\mathtt{DG}_0}(u_0, v_0) := B_{\mathtt{CDG}}(u_0, v_0) \qquad \forall u_0, v_0 \in V(\mathcal{T}_H, q).$$

Local Solvers, *i*=*1*,...,*N*

Prolongation (injection) operator $R_i^\top : V(\mathcal{T}_{h_i}, p) \to V(\mathcal{T}_h, p)$, where

$$\begin{aligned} V(\mathcal{T}_{h_i}, p) &= \{v \in L_2(\Omega_i) : v|_\kappa \in \mathcal{S}_{p_\kappa}(\kappa) \quad \forall \kappa \subset \Omega_i\}, \\ B_{\mathtt{DG}_i}(u_i, v_i) &:= B_{\mathtt{DG}}(R_i^\top u_i, R_i^\top v_i) \qquad \forall u_i, v_i \in V(\mathcal{T}_{h_i}, p). \end{aligned}$$

Schwarz Preconditioners for *hp*-DGFEM

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Coarse Solver (DGFEM)

$$B_{\mathtt{DG}_0}(u_0, v_0) := B_{\mathtt{CDG}}(u_0, v_0) \qquad \forall u_0, v_0 \in V(\mathcal{T}_H, q).$$

Local Solvers, *i*=1,…,*N*

Prolongation (injection) operator $R_i^\top : V(\mathcal{T}_{h_i}, p) \to V(\mathcal{T}_h, p)$, where

$$\begin{aligned} V(\mathcal{T}_{h_i}, p) &= \{v \in L_2(\Omega_i) : v|_\kappa \in \mathcal{S}_{p_\kappa}(\kappa) \quad \forall \kappa \subset \Omega_i\}, \\ B_{\mathtt{DG}_i}(u_i, v_i) &:= B_{\mathtt{DG}}(R_i^\top u_i, R_i^\top v_i) \qquad \forall u_i, v_i \in V(\mathcal{T}_{h_i}, p). \end{aligned}$$

Local Projection Operators

$\widetilde{P}_i : V(\mathcal{T}_h, p) \to V(\mathcal{T}_{h_i}, p)$:

$$B_{\mathtt{DG}_i}(\widetilde{P}_i u, v_i) := B_{\mathtt{DG}}(u, R_i^\top v_i) \quad \forall v_i \in V(\mathcal{T}_{h_i}, p).$$

$\widetilde{P}_0 : V(\mathcal{T}_h, p) \to V(\mathcal{T}_H, q)$:

$$B_{\mathtt{DG}_0}(\widetilde{P}_0 u, v_0) := B_{\mathtt{DG}}(u, R_0^\top v_0) \quad \forall v_0 \in V(\mathcal{T}_H, q).$$

Schwarz Operators

Writing $P_i := R_i^\top \widetilde{P}_i : V(\mathcal{T}_h, p) \to V(\mathcal{T}_h, p)$, for $i = 0, 1, \ldots, N$, we have

$$P_{\mathtt{ad}} := \sum_{i=0}^{N} P_i, \quad P_{\mathtt{mu}} := I - (I - P_N)(I - P_{N-1}) \cdots (I - P_0).$$

Schwarz Operators

Writing $P_i := R_i^\top \widetilde{P}_i : V(\mathcal{T}_h,p) \to V(\mathcal{T}_h,p)$, for $i = 0,1,\ldots,N$, we have

$$P_{\mathtt{ad}} := \sum_{i=0}^{N} P_i, \qquad P_{\mathtt{mu}} := I - (I - P_N)(I - P_{N-1}) \cdots (I - P_0).$$

Algebraic Formulation for Additive Schwarz

$$\begin{aligned} \tilde{P}_i &= A_i^{-1} R_i A, \\ P_i &:= R_i^\top \tilde{P}_i = R_i^\top A_i^{-1} R_i A, \\ P_{\mathtt{ad}} &= \left(\sum_{i=0}^{N} R_i^\top A_i^{-1} R_i \right) A. \end{aligned}$$

- A: Full DGFEM matrix.
- A_i, $i > 1$: Local DGFEM matrix on Ω_i.
- A_0: Composite DGFEM matrix.
- $R_i : V(\mathcal{T}_h,p) \to V(\mathcal{T}_{h_i},p)$: Restriction.
- $R_i^\top : V(\mathcal{T}_{h_i},p) \to V(\mathcal{T}_h,p)$: Prolongation.
- $P_{\mathtt{ad}}$: Preconditioned system.

Theorem

The condition number $\kappa(P_{\text{ad}})$ is bounded by:

$$\kappa(P_{\text{ad}}) \leq C\gamma \frac{p^2}{q} \frac{H}{h}.$$

- For details, see: Antonietti & H. 2011, Antonietti, Giani, & H. 2013, Antonietti, H., & Smears 2015.
- Proof is based on the abstract theory of Schwarz methods, cf. Dryja & Widlund, 1989, 1990, and standard arguments for *hp*-DGFEMs.
- Scalability (i.e., independent of the number of subdomains).
- Note: ***No overlap*** is required unlike with CGFEM

Poisson's Equation

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Domain with 4 holes

$h \backslash H$	1/2	1/4	1/8	1/16	1/32	1/64
1/8	**32 (42.1)**	**27 (14.5)**	-	-	-	-
1/16	**58 (96.8)**	**47 (40.1)**	29 (17.5)	-	-	-
1/32	**93 (203.2)**	**74 (89.8)**	**48 (44.1)**	31 (17.8)	-	-
1/64	**134 (411.2)**	**121 (188.3)**	80 (95.4)	**50 (44.2)**	31 (17.9)	-
1/128	**192 (821.9)**	**185 (369.8)**	137 (194.3)	80 (95.2)	**50 (44.2)**	31(17.9)

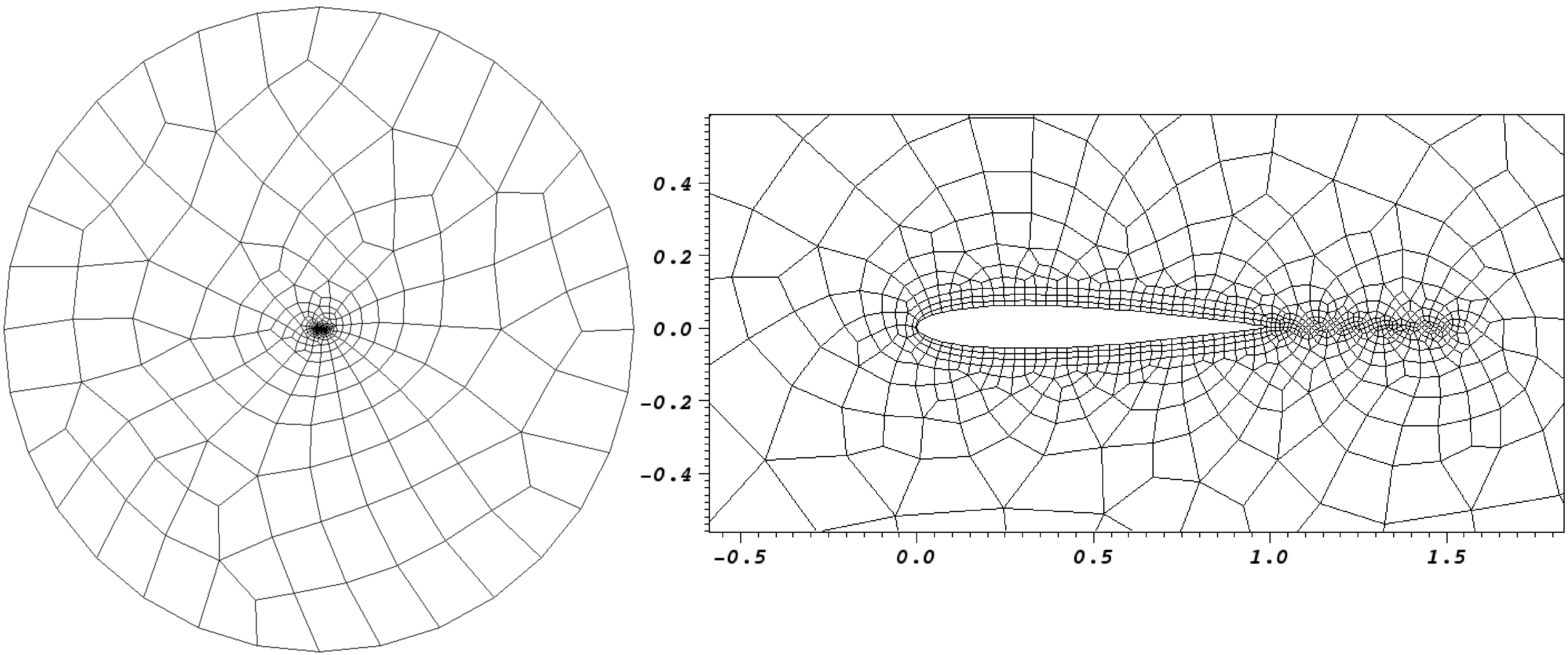
Domain with 4 holes

$h \backslash H$	1/2	1/4	1/8	1/16	1/32	1/64
1/8	**32 (42.1)**	**27 (14.5)**	-	-	-	-
1/16	**58 (96.8)**	**47 (40.1)**	29 (17.5)	-	-	-
1/32	**93 (203.2)**	**74 (89.8)**	**48 (44.1)**	31 (17.8)	-	-
1/64	**134 (411.2)**	**121 (188.3)**	80 (95.4)	**50 (44.2)**	31 (17.9)	-
1/128	**192 (821.9)**	**185 (369.8)**	137 (194.3)	80 (95.2)	**50 (44.2)**	31(17.9)

Domain with 256 holes

$h \backslash H$	1/2	1/4	1/8	1/16	1/32	1/64
1/64	**55 (83.8)**	**55 (81.3)**	**54 (69.2)**	**50 (40.4)**	**31 (14.7)**	-
1/128	**79 (178.6)**	**79 (174.5)**	**79 (151.4)**	**76 (93.2)**	**52 (38.2)**	31 (17.6)

Ma=0.5, Re=5000, $\alpha = 2^\circ$ and adiabatic wall condition

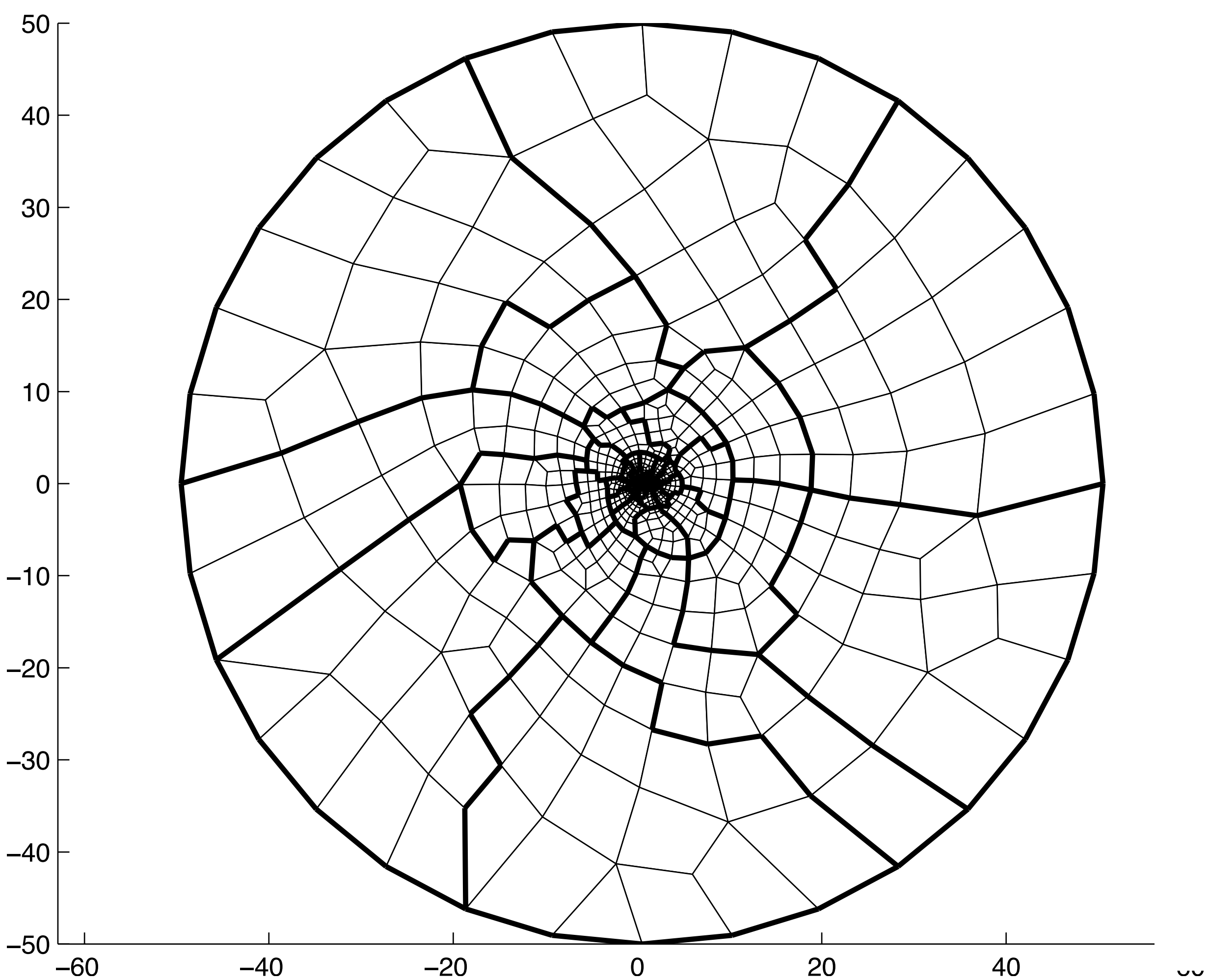


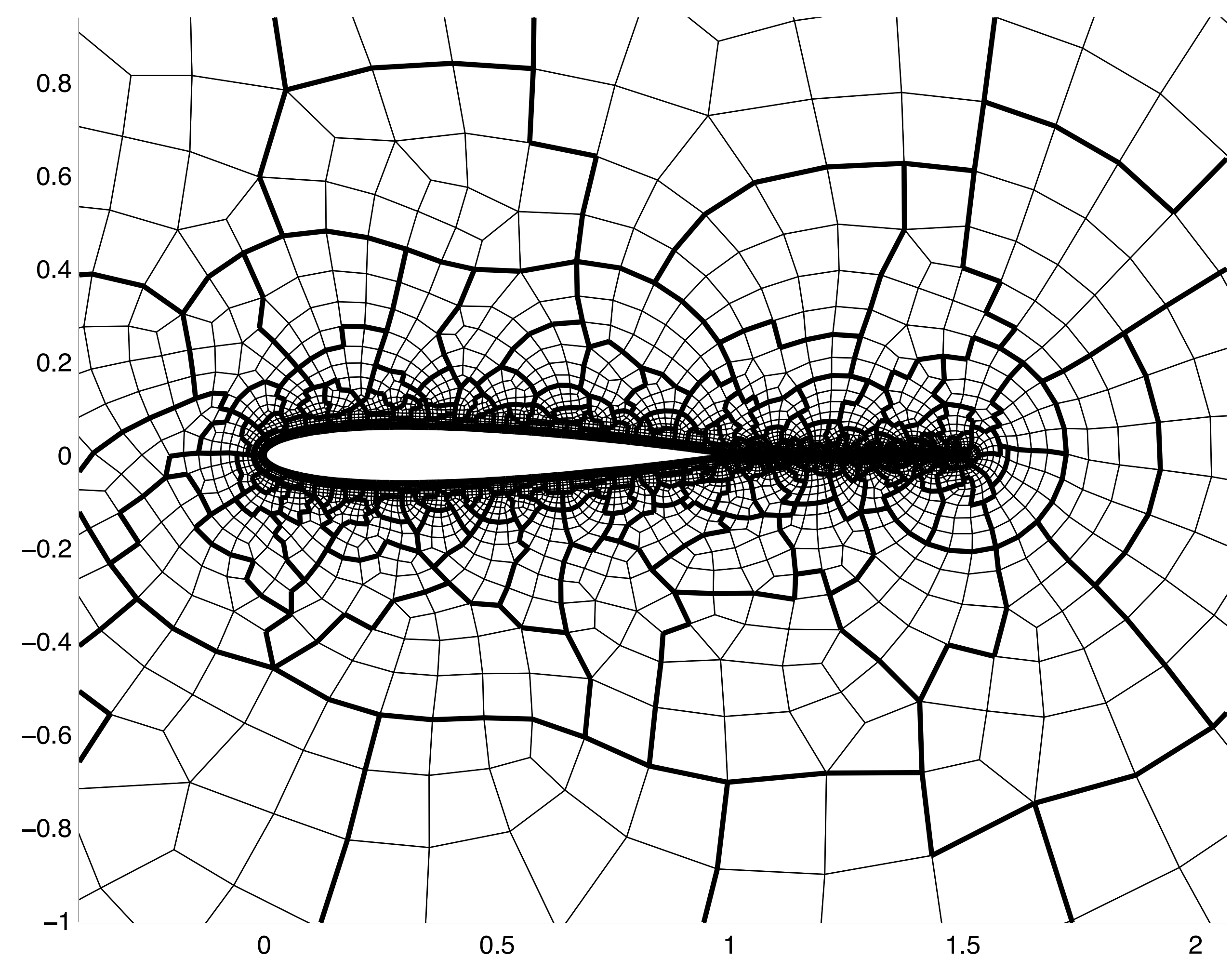
Mesh 1, consisting of 578 (hybrid) elements

2D Laminar Flow: NACA0012 Airfoil

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

METIS is employed to generate both $\mathcal{T}_S$ with $N = 250$ and $\mathcal{T}_H$.





Mesh 5 partitioned into 500 regions using METIS

2D Laminar Flow: NACA0012 Airfoil

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Ma=0.5, Re=5000, $\alpha = 2^\circ$ and adiabatic wall condition

$\mathcal{T}_h \setminus$ # Eles $\mathcal{T}_H$	**500**	**1000**	**2000**	**4000**	**8000**
Mesh 2	**124 (936,10)**	-	-	-	-
Mesh 3	186 (1303,9)	**121 (800,9)**	-	-	-
Mesh 4	310 (1957,9)	168 (1150,9)	**116 (700,9)**	-	-
Mesh 5	519 (3136,9)	278 (1796,9)	151 (1034,9)	**95 (646,9)**	-
Mesh 6	933 (5604,9)	492 (3034,9)	276 (1785,9)	162 (1090,9)	**103 (687,9)**

METIS is employed to generate both $\mathcal{T}_S$ with $N = 250$ and $\mathcal{T}_H$.

Meshes 2-6: 1134, 2113, 4246, 8946, 20229 elements, respectively.

Summary and Outlook

Summary and Outlook

- Developed the error analysis of DGFEMs on general polytopic meshes:
  - ☑ Number of degrees of freedom is *independent* of the domain;
  - ☑ Coarse approximations may be computed with engineering accuracy;
  - ☑ Adaptivity is focused on resolving *important features* of the solution;
  - ☑ Method naturally admits high-order polynomial orders;
  - ☑ May be exploited as coarse level solvers with multilevel solvers.
    DD: Antonietti, Giani, & H. 2014, Giani & H. 2014, Antonietti, H., & Smears 2015.
- Analysis of DGFEMs on general polygonal/polyhedral meshes accounts for local edge/face degeneration.

Summary and Outlook

- Developed the error analysis of DGFEMs on general polytopic meshes:
  - ☑ Number of degrees of freedom is *independent* of the domain;
  - ☑ Coarse approximations may be computed with engineering accuracy;
  - ☑ Adaptivity is focused on resolving *important features* of the solution;
  - ☑ Method naturally admits high-order polynomial orders;
  - ☑ May be exploited as coarse level solvers with multilevel solvers.
    DD: Antonietti, Giani, & H. 2014, Giani & H. 2014, Antonietti, H., & Smears 2015.
- Analysis of DGFEMs on general polygonal/polyhedral meshes accounts for local edge/face degeneration.
- Development of multigrid solvers.
  Antonietti, H., Sarti, & Verani 2014
- Extension to problems with discontinuous coefficients.
- Application to two-grid methods for nonlinear PDEs.
  Congreve, H., & Wihler 2011, 2013, Congreve & H. 2013
- Efficient Quadrature.
- A posteriori error estimation.