Adaptivity and blowup detection for nonlinear evolution PDEs

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Based on joint work with:
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Overview

- Motivation – blowup detection
- Rigorous a posteriori bounds for semilinear parabolic problems, valid up to blowup time
- Adaptivity and estimation of blowup time and near blowup

A. Cangiani, E. H. Georgoulis, I. Kyza, S. Metcalfe, *Adaptivity and blow-up detection for nonlinear evolution problems*, in review.

- adaptive (high-order) methods for non-polygonal interface problems

A. Cangiani, E. H. Georgoulis, Y. Sabawi, *Adaptive discontinuous Galerkin methods for non-polygonal interface problems*, in preparation.

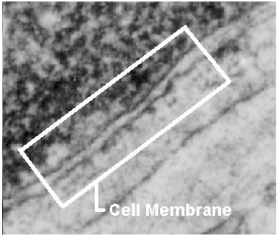
A "simple" test case

$u_t - \Delta u = u^2, \qquad u(0,x)$ Gaussian $\qquad \rightsquigarrow$ single point blowup

Motivation

- blowup time estimation is interesting for applications (physical/chemical reactions, chemotaxis(?), etc.)
- a priori/analytical knowledge of blowup times is known for few model problems only
- a general enough error estimation framework could provide insight for interaction of non-linearities with other phenomena, such as advection, interfaces,etc.

Mass transfer through semi-permeable membranes



We consider a number of solutes subject to:

- diffusion and advection on both sides of the membrane
- nonlinear reactions with other solutes
- mass transfer across the membrane

(red = linear, green = nonlinear)

Cargo·Imp concentration in cell signal transduction

Simpler model problem...

Consider advection-diffusion-reaction PDE problem on a single domain $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} \partial_t u - \kappa \Delta u + \mathbf{a} \cdot \nabla u + f(u) &= 0 && \text{in } \Omega \times (0, T], \\ u &= 0 && \text{on } \partial\Omega \times (0, T], \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

with $f(u) = -u^2$, $\kappa > 0$.

- analytical results on blowup times ?
- effect(s) of advection w.r.t. to blowup?
- inclusion of interfaces in the mix? (ongoing...)

For accessibility, in this talk, I shall mostly discuss the even simpler problem:

$$\begin{aligned} \partial_t u - \Delta u &= u^2 && \text{in } \Omega \times (0, T], \\ u &= 0 && \text{on } \partial\Omega \times (0, T], \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

Aim

Estimation of blowup time & *space-time error control* near blowup

Blowup detection & error control

A case for adaptivity:

- Extremely fine (space-time) local resolution needed to approach blowup time
- "Standard" a priori error analysis & uniform meshes:
  at time $t = T_{blowup} - \varepsilon$, constants $\mathrm{O}(\mathrm{e}^{1/\varepsilon})$ appear in bounds. $\varepsilon \to 0$...
- Problem of missing the blowup!

Dominant approach in the literature: rescaling/use of PDE 'similarity' properties to (r-)adapt/rescale discretisation parameters. Nakagawa ('75), Berger & Kohn ('89), Stuart & Floater ('90), Tourigny & Sanz-Serna ('92), Budd, Huang & Russell ('96) ...

Approach

Construction of adaptive algorithms via rigorous *a posteriori error estimates*

- Limited literature on a posteriori error control & adaptivity in this context.
  Karakashian & Plexousakis ('96), Kyza & Makridakis ('11)
- Conditional a posteriori error estimates: final estimates hold under some computationally verifiable conditions

Step back to ODEs...

ODE initial value problem: find $u : [0, T] \to \mathbb{R}$ such that

$$\begin{aligned} \frac{du}{dt} &= f(u) := u^p, \quad \text{in } (0, T], \\ u(0) &= u_0, \end{aligned}$$

with $\mathbb{N} \ni p \geq 2$, so that the solution blows up in finite time, say T^*.

Three different one step schemes: set $U^0 := u_0$; for $k = 1, \ldots, N$, solve for U^k:

$$\frac{U^k - U^{k-1}}{\tau_k} = F(U^{k-1}, U^k),$$

with F one of the following three classical approximations of f:

$$\begin{aligned} &\text{Explicit Euler} && F(U^{k-1}, U^k) = f(U^{k-1}), \\ &\text{Implicit Euler} && F(U^{k-1}, U^k) = f(U^k), \\ &\text{RK2/Improved Euler} && F(U^{k-1}, U^k) = \frac{1}{2}\left(f(U^{k-1}) + f\left(U^{k-1} + \tau_k f(U^{k-1})\right)\right). \end{aligned}$$

An a posteriori error estimate

Let $U : [0, T] \to \mathbb{R}$ p/w linear interpolant of $\{U^k\}$ at t^k, viz

$$U(t) := \ell_{k-1}(t)U^{k-1} + \ell_k(t)U^k, \qquad t \in (t^{k-1}, t^k],$$

Hence, on each interval $(t^{k-1}, t^k]$, we have

$$\frac{dU}{dt} = F(U^{k-1}, U^k).$$

Therefore, on each time interval $(t^{k-1}, t^k]$, the error $e := u - U$ satisfies

$$\frac{de}{dt} = f(u) - F(U^{k-1}, U^k) = f(U) + f'(u)e + \sum_{j=2}^{p} \frac{f^{(j)}(U)}{j!} e^j - F(U^{k-1}, U^k),$$

or, setting $\eta_k := f(U) - F(U^{k-1}, U^k)$, we have

$$\frac{de}{dt} = \eta_k + \Big(f'(U) + \sum_{j=2}^{p} \frac{f^{(j)}(U)}{j!} e^{j-1}\Big)e.$$

An a posteriori error estimate

Gronwall's inequality, therefore, implies

$$|e(t)| \le H_k(t) G_k \phi_k,$$

$$\text{with} \quad H_k(t) := \exp\Big(\sum_{j=2}^{p} \int_{t^{k-1}}^{t} \frac{|f^{(j)}(U)|}{j!} |e|^{j-1} \,\mathrm{d}s\Big), \quad G_k := \exp\Big(\int_{t^{k-1}}^{t^k} |f'(U)| \,\mathrm{d}s\Big),$$
$$\text{and} \quad \phi_k := |e(t^{k-1})| + \int_{t^{k-1}}^{t^k} |\eta_k| \,\mathrm{d}s.$$

Theorem (Conditional error estimate)

For $k = 1, \ldots, N$, the following a posteriori estimate holds:

$$\max_{t \in [t^{k-1}, t^k]} |e(t)| \le \delta_k G_k \phi_k,$$

provided that $\delta_k > 1$ is chosen so that

$$\sum_{j=2}^{p} (\delta_k G_k \phi_k)^{j-1} \int_{t^{k-1}}^{t^k} \frac{|f^{(j)}(U(s))|}{j!} \,\mathrm{d}s - \log(\delta_k) = 0.$$

Two algorithms: 1

Algorithm 1 ODE Algorithm 1

Input: f, F, u_0, τ_1, `tol`.
Compute U^1 from U^0.
while $\int_{t^0}^{t^1} |\eta_1| \, \mathrm{d}s > \texttt{tol}$ **do**
  $\tau_1 \leftarrow \tau_1/2$.
  Compute U^1 from U^0.
end while
Compute δ_1.
Set $k = 0$.
while δ_{k+1} exists **do**
  $k \leftarrow k + 1$.
  $\tau_{k+1} = \tau_k$.
  Compute U^{k+1} from U^k.
  while $\int_{t^k}^{t^{k+1}} |\eta_{k+1}| \, \mathrm{d}s > \texttt{tol}$ **do**
    $\tau_{k+1} \leftarrow \tau_{k+1}/2$.
    Compute U^{k+1} from U^k.
  end while
  Compute δ_{k+1}.
end while
Output: k, t^k.

$$\eta_k := f(U) - F(U^{k-1}, U^k)$$

Absolute tolerance: `tol`

$$|T_{blowup} - T_{final}| \sim N^{-r}$$

Method	$p = 2$	$p = 3$
Implicit Euler	$r \approx 0.66$	$r \approx 0.79$
Explicit Euler	$r \approx 1.35$	$r \approx 1.60$
Improved Euler	$r \approx 1.2$	$r \approx 1.48$

Two algorithms: 2

Algorithm 2 ODE Algorithm 2

Input: f, F, u_0, τ_1, $\mathtt{tol}$.
Compute U^1 from U^0.
while $\int_{t^0}^{t^1} |\eta_1| \, \mathrm{d}s > \mathtt{tol}$ **do**
  $\tau_1 \leftarrow \tau_1/2$.
  Compute U^1 from U^0.
end while
Compute δ_1.
$\mathtt{tol} = G_1 * \mathtt{tol}$.
Set $k = 0$.
while δ_{k+1} exists **do**
  $k \leftarrow k + 1$.
  $\tau_{k+1} = \tau_k$.
  Compute U^{k+1} from U^k.
  while $\int_{t^k}^{t^{k+1}} |\eta_{k+1}| \, \mathrm{d}s > \mathtt{tol}$ **do**
    $\tau_{k+1} \leftarrow \tau_{k+1}/2$.
    Compute U^{k+1} from U^k.
  end while
  Compute δ_{k+1}.
  $\mathtt{tol} = G_{k+1} * \mathtt{tol}$.
end while
Output: k, t^k.

$$\eta_k := f(U) - F(U^{k-1}, U^k)$$

Relative tolerance: $G_{k+1} * \mathtt{tol}$

$$|T_{blowup} - T_{final}| \sim N^{-r}$$

Method	$p = 2$	$p = 3$
Implicit Euler	$r \approx 1.00$	$r \approx 1.00$
Explicit Euler	$r \approx 1.45$	$r \approx 1.43$
Improved Euler	$r \approx 2.03$	$r \approx 2.03$

Back to PDEs: time semi-discretisation

$0 := t_0 < t_1 < \cdots < t_N =: T$ partition of $[0, T]$, $\tau_k := t_{k+1} - t_k$,

Implicit–Explicit (IMEX) Euler method: find $U^k \in H_0^1(\Omega)$, $k = 0, 1, \ldots, N-1$:

$$\frac{U^{k+1} - U^k}{\tau_k} - \Delta U^{k+1} = f(U^k), \text{ with } U^0 = u_0$$

Why IMEX

- implicit on diffusion $\Rightarrow$ stability
- explicit on nonlinear reaction $\Rightarrow$ advantageous approximation near blowup

Error equation

Let $U : [0, T] \to H_0^1(\Omega)$ linear interpolant of $\{U^k\}_k$.

Let $e := u - U$. Then, for $f(u) = u^2$, we have

$$\partial_t e - \Delta e = 2Ue + e^2 + r_{k+1}$$

in $(t_k, t_{k+1}]$ with $r_{k+1} := \left[f(U) - f(U^k)\right] + (t_{k+1} - t)(U^{k+1} - U^k)/\tau_k$.

Energy estimate:

$$\frac{d}{dt}\|e(t)\|^2 + \|\nabla e(t)\|^2 \leq 4\|U(t)\|_{L^\infty}\|e(t)\|^2 + 2\langle e^2, e\rangle + \|r_{k+1}(t)\|_{-1}^2$$

Using $\langle e^2, e\rangle \leq \|e\|_{L^\infty}\|e\|^2$, Gronwall's inequality gives

$$\max_{0\leq t\leq T}\|e(t)\|^2 \leq \exp\Big(2\int_0^T [2\|U(t)\|_{L^\infty} + \|e(t)\|_{L^\infty}]\,\mathrm{d}t\Big)\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\|r_{k+1}(t)\|_{-1}^2\,\mathrm{d}t$$

- Is a fully a posteriori bound possible from this?
- Behaviour of the constant in the run up to blowup?

Exponent growth (Kyza & Makridakis ('11))

Gronwall exponent:

$$\int_0^T 2\|U(t)\|_{L^\infty} + \|e(t)\|_{L^\infty}\, dt$$

On run up to blowup time: $||u(t)||_{L^\infty} \sim \dfrac{1}{(T_{blowup} - t)}$ (Merle & Zaag ('00))

We can infer that we also have, approximately,

$$||U(t)||_{L^\infty},\ \|e(t)\|_{L^\infty} \sim \frac{1}{(T_{blowup} - t)}$$

Hence, at $T := T_{blowup} - \varepsilon$, $\varepsilon > 0$, the exponent scales like

$$\int_0^{T_{blowup}-\varepsilon} \frac{1}{T_{blowup} - t}\, dt = \ln\left(\frac{T_{blowup}}{\varepsilon}\right)$$

and, thus, the Gronwall constant scales like

$$\exp\left(\int_0^{T_{blowup}-\varepsilon} \frac{1}{T_{blowup} - t}\, dt\right) \sim \frac{C(T_{blowup})}{\varepsilon^q}$$

i.e., polynomial growth w.r.t ε! Challenge: Leads to practical algorithm?

Time-discrete scheme a posteriori bound of (Kyza & Makridakis ('11))

fixed point arguments + semigroup theory $\Longrightarrow$ a posteriori estimate

Conditional a posteriori error estimates

- $L^\infty(L^\infty)$: $\|e\|_{L^\infty(L^\infty)} \leq \mathrm{e}^{1/8+4\int_0^T \|U(s)\|_{L^\infty}\,\mathrm{d}s} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \|r_{k+1}(s)\|_{L^\infty} \mathrm{d}s$
- $L^\infty(L^2)$: $\max_{0\leq t\leq T} \|e(t)\|^2 \leq \mathrm{e}^{1/8+4\int_0^T \|U(s)\|_{L^\infty}\,\mathrm{d}s} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \|r_{k+1}(s)\|_{-1}^2 \mathrm{d}s$

provided τ_k are chosen so that

$$\mathrm{e}^{4\int_0^T ||U(s)||_{L^\infty}\,\mathrm{d}s} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \|r_{k+1}(s)\|_{L^\infty} \mathrm{d}s \leq \frac{3}{16}\rho, \quad \text{with } \rho < \frac{1}{16T}$$

Global condition! efficient time-adaptive algorithm? – More local conditions?

- For fully-discrete L^∞-norm a posteriori bounds for resp. elliptic problem are needed in this framework.
- growth range u^p, $p > 1$.

A new a posteriori bound (fully discrete bound also available)

New conditional estimate

$$\max_{t_k \le t \le t_{k+1}} \|e(t)\|^2 \le \delta_{k+1} \mathrm{e}^{4\int_{t_k}^{t_{k+1}} \|U(s)\|_{L^\infty} \, \mathrm{d}s} \left[\|e(t_k)\|^2 + \int_{t_k}^{t_{k+1}} \|r_{k+1}(s)\|_{-1}^2 \, \mathrm{d}s \right]$$

with $e(0) = 0$, where $\delta_{k+1} > 1$ is chosen to satisfy

$$\beta^2 \delta_{k+1} \mathrm{e}^{4\int_{t_k}^{t_{k+1}} \|U(s)\|_{L^\infty} \, \mathrm{d}s} \left[\|e(t_k)\|^2 + \int_{t_k}^{t_{k+1}} \|r_{k+1}(s)\|_{-1}^2 \, \mathrm{d}s \right] \tau_k - \ln \delta_{k+1} = 0$$

Ingredients:

- Gagliardo–Nirenberg inequality: $\langle e^2, e \rangle(t) \le \beta \|e(t)\|^2 \|\nabla e(t)\|$
- a new local on each time-step continuation argument

  ("global" continuation arguments in this spirit: Kessler, Nochetto & Schmidt ('04), Bartels ('05), Cangiani, G. & Jensen ('13))

Condition – discussion

Condition satisfied only if

$$\beta^2 \mathrm{e}^{4\int_{t_k}^{t_{k+1}} \|U(s)\|_{L^\infty}\,\mathrm{d}s} \left[\|e(t_k)\|^2 + \int_{t_k}^{t_{k+1}} \|r_{k+1}(s)\|_{-1}^2 \,\mathrm{d}s \right] \tau_k < \frac{1}{\mathrm{e}}$$

This implies *restriction on time-steps τ_k*, i.e., conditional estimates.

Is condition practical?

- Say $t_{k+1} = T_{blowup} - \varepsilon$, then $\mathrm{e}^{4\int_{t_k}^{t_{k+1}} \|U(s)\|_{L^\infty}\,\mathrm{d}s} \sim \left(\frac{\tau_k + \varepsilon}{\varepsilon}\right)^q$
- $\int_{t_n}^{t_{n+1}} \|r_{n+1}(s)\|_{-1}^2 \,\mathrm{d}s$ estimable via a posteriori bounds

δ_{n+1} computed via, e.g., Newton's method.

Semilinear advection-diffusion problem with blowup

$$
\begin{aligned}
u_t - \epsilon\Delta u + \mathbf{a}\cdot\nabla u &= u^2 + g && \text{in } \Omega\times(0,T],\\
u &= 0 && \text{on } \partial\Omega\times(0,T],\\
u(\cdot,0) &= u_0 && \text{in } \Omega,
\end{aligned}
$$

Spatial discretisation: Discontinuous Galerkin (dG) method with upwind flux.

dG IMEX method: find $U_h^{k+1}\in\mathbb{V}_h^{k+1}$ s.t.

$$
\langle\frac{U_h^{k+1}-U_h^k}{\tau_k}, V_h\rangle + B(t_{k+1};U_h^{k+1},V_h) = \langle f(U_h^k)+g^k, V_h\rangle \quad \forall V_h\in\mathbb{V}_h^{k+1}
$$

Continuation Argument + Elliptic Reconstruction

$\Downarrow$

Conditional a posteriori error estimate in the $L^\infty(L^2)-$norm for dG IMEX

For elliptic reconstruction see Makridakis & Nochetto ('03), Lakkis & Makridakis ('06), G. Lakkis & Virtanen ('11), Cangiani, G., & Metcalfe ('14)

Key attribute of the approach: Flexibility on the elliptic operator!

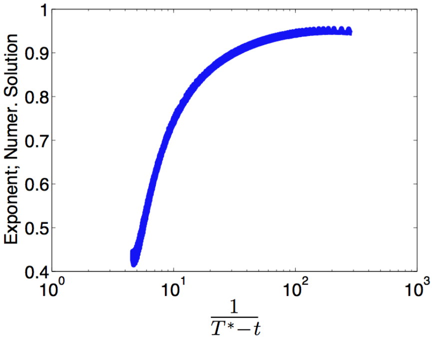
A simple test case: reaction-diffusion problem

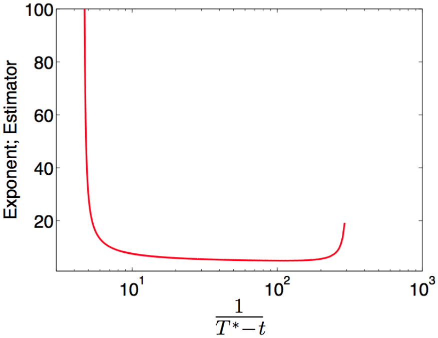
$\Omega = (-4,4) \times (-4,4)$, $u_0(x,y) = 10\mathrm{e}^{-2(x^2+y^2)}$ $\rightsquigarrow$ single point blowup

- space-time adaptive algorithm
- when condition fails, restarts with smaller timestep

A simple test case

tol	# time-steps	T	$\|U_h(T)\|_{L^\infty}$	Estimator
$(0.125)^{10}$	6956	0.21228	238.705	33426.7
$(0.125)^{11}$	14008	0.21375	343.078	36375.0
$(0.125)^{12}$	28151	0.21478	496.885	66012.8
$(0.125)^{13}$	35580	0.21549	722.884	157300.0





$\|U_h(t)\|_{L^\infty} \sim \dfrac{1}{(T^*-t)^{p_k}}$ plots depict: p_k vs. $\dfrac{1}{T^*-t}$

Observed rates of convergence

- $\|U_h(t)\|_{L^\infty}$ appears to blow up at the expected rate on the run-up to T^*
- we have $|T^* - T| \sim N^{-1/2}$
  - shortcoming of energy method?
  - semigroup techniques?
- Estimator blows up at faster rate, but delivers optimal blowup rate for the numerical solution regardless!

Numerical experiment – advection-diffusion problem

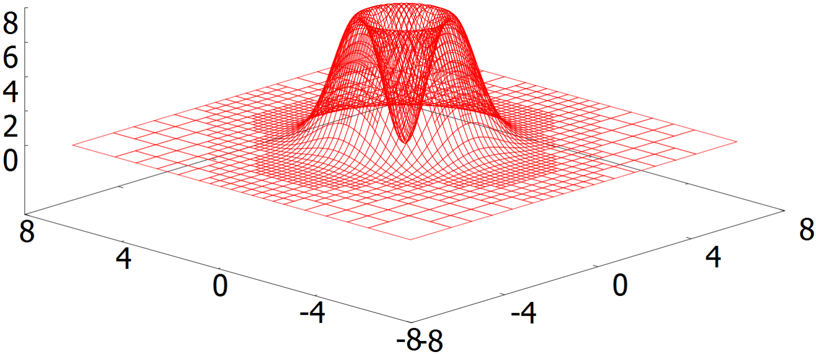
$\Omega = (-4,4) \times (-4,4)$, $\kappa = 1$, $\mathbf{a} \equiv (1,1)^{\mathrm{T}}$, $g \equiv 1$, $u_0 \equiv 0$

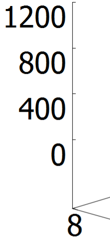
`tol`	# time-steps	T	$\|U_h(T)\|_{L^\infty}$
1	4	0.78125	0.886245
0.125	10	0.976562	1.32178
$(0.125)^2$	54	1.31836	3.26904
$(0.125)^3$	119	1.41602	5.10672
$(0.125)^4$	252	1.48163	8.05863
$(0.125)^5$	520	1.51711	11.8193
$(0.125)^6$	1064	1.54467	18.1385
$(0.125)^7$	2158	1.56224	27.4045
$(0.125)^8$	4354	1.57402	41.3737
$(0.125)^9$	8792	1.58243	64.4503
$(0.125)^{10}$	17713	1.58770	99.1902
$(0.125)^{11}$	35580	1.59092	145.785
$(0.125)^{12}$	71352	1.59299	211.278

Numerical experiment – blowup on 1D manifold

$\Omega = (-8, 8)^2$, $\kappa = 1$, $\mathbf{a} = (0, 0)^T$, $f_0 = 0$

‘volcano’ type initial condition be given by $u_0 = 10(x^2 + y^2)\exp(-(x^2 + y^2)/2)$





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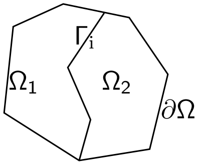
`ttol`$^+$	Time Steps	Estimator	Final Time	$\|\|U_h(T)\|\|_{L^\infty(\Omega)}$
8	3	15	0.06250	10.371
1	10	63	0.09375	14.194
0.125	36	211	0.11979	21.842
$(0.125)^2$	86	533	0.13412	31.446
$(0.125)^3$	190	971	0.14388	45.122
$(0.125)^4$	404	1358	0.15072	64.907
$(0.125)^5$	880	5853	0.15601	98.048
$(0.125)^6$	1853	10654	0.15942	146.162
$(0.125)^7$	3831	21301	0.16176	219.423
$(0.125)^8$	7851	143989	0.16336	332.849
$(0.125)^9$	16137	287420	0.16442	505.236
$(0.125)^{10}$	32846	331848	0.16512	769.652
$(0.125)^{11}$	66442	626522	0.16558	1175.21

Notation

Consider an open polygonal domain $\Omega \subset \mathbb{R}^d$ subdivided into two subdomains Ω_1 and Ω_2:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_\mathrm{i}$$

$$\Gamma_\mathrm{i} = \bar{\Omega}_1 \cap \bar{\Omega}_2$$



and set $\mathcal{H}^1 := [H^1(\Omega_1 \cup \Omega_2)]^n$, $n \in \mathbb{N}$.
PDE system:

$$\begin{aligned}
\mathbf{u}_t - \nabla \cdot (A\nabla \mathbf{u} - U\mathbf{B}) + \mathbf{F}(\mathbf{u}) = \mathbf{0} \quad & \text{in } (0, T] \times (\Omega_1 \cup \Omega_2), \\
\mathbf{u}(0, x) = \mathbf{u}_0(x) \quad & \text{on } \{0\} \times \Omega, \\
\mathbf{u} = \mathbf{g}_\mathrm{D} \quad & \text{on } \Gamma_\mathrm{D}, \\
\big(A\nabla \mathbf{u} - \chi^- U\mathbf{B}\big)\mathbf{n} = \mathbf{g}_\mathrm{N} \quad & \text{on } \Gamma_\mathrm{N},
\end{aligned}$$

where χ^- the (vector-valued) characteristic function of the inflow part of $\partial\Omega$ and $U := \mathrm{diag}(\mathbf{u})$. On Γ_i we impose:

$$(A\nabla \mathbf{u} - U\mathbf{B})\mathbf{n}|_{\Omega^1} = \mathbf{P}(\mathbf{u})(\mathbf{u}^2 - \mathbf{u}^1) - \{U\}_w R\mathbf{B}\mathbf{n}^1$$

$$(A\nabla \mathbf{u} - U\mathbf{B})\mathbf{n}|_{\Omega^2} = \mathbf{P}(\mathbf{u})(\mathbf{u}^1 - \mathbf{u}^2) - \{U\}_w R\mathbf{B}\mathbf{n}^2$$

Reflective membranes – time-dependent sharp features

uniform

adaptive

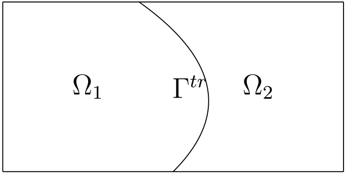
- a DG method & a priori analysis with solution boundedness assumption

  Cangiani, G., Jensen ('13)

- No analytical information – numerics?

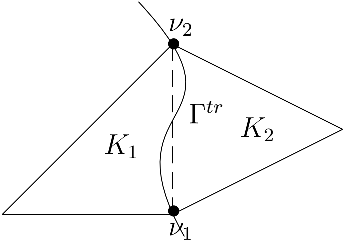
A posteriori error bounds for dG on curved geometries

$\Omega \subset \mathbb{R}^d$, $d = 2, 3$. $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma^{tr}$, with $\Gamma^{tr} := (\partial\Omega_1 \cap \partial\Omega_2) \setminus \partial\Omega$ Lipschitz.



$$
\begin{aligned}
-\Delta u &= f, && \text{in } \Omega_1 \cup \Omega_2,\\
u &= 0, && \text{on } \partial\Omega,\\
\mathbf{n}^1 \cdot \nabla u_1 &= C_{tr}(u_2 - u_1)|_{\Omega_1} && \text{on } \bar{\Omega}_1 \cap \Gamma^{tr},\\
\mathbf{n}^2 \cdot \nabla u_2 &= C_{tr}(u_1 - u_2)|_{\Omega_2} && \text{on } \bar{\Omega}_2 \cap \Gamma^{tr},
\end{aligned}
$$

where $u_i = u|_{\bar{\Omega}_i \cap \Gamma^{tr}}$, $i = 1, 2$, C_{tr} a given permeability constant.



Conclusions

- 'Energy methods' for error control are relevant and competitive for nonlinear evolution problems
- Error control for curved interfaces/boundaries

Some references

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Computation of numerical blowup time and blowup rate

Let t^* denote a numerical blowup time. We implement a set numerical experiments (corresponding to different tolerances) producing U_h^ℓ, $\ell = 1, \cdots L$ approximations to the exact solution u. Assume that

$$\|U_h^\ell(t_n)\|_{L^\infty} \sim \left(\frac{1}{t^* - t_n}\right)^p$$

1. Since $f(u) = u^2$, assume that $p = 1$, and use $\|U_h^L(t_{N-1})\|_{L^\infty}$, $\|U_h^L(T)\|_{L^\infty}$ to calculate

$$\left.\begin{aligned} \|U_h^L(t_{N-1})\|_{L^\infty} &= C_L \frac{1}{t^* - t_{N-1}} \\ \|U_h^L(T)\|_{L^\infty} &= C_L \frac{1}{t^* - T} \end{aligned}\right\} \Rightarrow t^* = \frac{T\|U_h^L(T)\|_{L^\infty} - t_{N-1}\|U_h^L(t_{N-1})\|_{L^\infty}}{\|U_h^L(t_{N-1})\|_{L^\infty} - \|U_h^L(t_{N-1})\|_{L^\infty}}$$

- For the considered example, $t^* = 0.21705$

2. Consider $t^*(= 0.21705)$ as the numerical blowup time. We use $\|U_h^\ell(t)\|_{L^\infty}$, $\ell \neq L$ to compute the numerical blowup time:

$$p_n := \frac{\ln\left(\|U_h^\ell(t_{n+1})\|_{L^\infty} / \|U_h^\ell(t_n)\|_{L^\infty}\right)}{\ln\left((t^* - t_n)/(t^* - t_{n+1})\right)}$$

- We expect $p_n \to 1$ as $n \to N$, for the considered model problem