Adaptivity and blowup detection for nonlinear evolution PDEs

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Based on joint work with:
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Overview

- Motivation – blowup detection

- Rigorous a posteriori bounds for semilinear parabolic problems, valid up to blowup time

- Adaptivity and estimation of blowup time and near blowup


- adaptive (high-order) methods for non-polygonal interface problems

A “simple” test case

$$u_t - \Delta u = u^2, \quad u(0, x) \text{ Gaussian} \quad \leadsto \text{single point blowup}$$
Motivation

- blowup time estimation is interesting for applications (physical/chemical reactions, chemotaxis(?), etc.)

- a priori/analytical knowledge of blowup times is known for few model problems only

- a general enough error estimation framework could provide insight for interaction of non-linearities with other phenomena, such as advection, interfaces, etc.
We consider a number of solutes subject to:

- **diffusion** and **advection** on both sides of the membrane
- **nonlinear reactions** with other solutes
- **mass transfer** across the membrane

(red = linear, green = nonlinear)
Cargo·Imp concentration in cell signal transduction
Simpler model problem...

Consider advection-diffusion-reaction PDE problem on a single domain $\Omega \subset \mathbb{R}^2$:

$$\partial_t u - \kappa \Delta u + a \cdot \nabla u + f(u) = 0 \quad \text{in } \Omega \times (0, T],$$

$$u = 0 \quad \text{on } \partial \Omega \times (0, T],$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega,$$

with $f(u) = -u^2$, $\kappa > 0$.

- analytical results on blowup times?
- effect(s) of advection w.r.t. to blowup?
- inclusion of interfaces in the mix? (ongoing...)

For accessibility, in this talk, I shall mostly discuss the even simpler problem:

$$\partial_t u - \Delta u = u^2 \quad \text{in } \Omega \times (0, T],$$

$$u = 0 \quad \text{on } \partial \Omega \times (0, T],$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega,$$

Aim

Estimation of blowup time & space-time error control near blowup
Blowup detection & error control

A case for adaptivity:
- Extremely fine (space-time) local resolution needed to approach blowup time
- “Standard” a priori error analysis & uniform meshes:
  at time $t = T_{\text{blowup}} - \varepsilon$, constants $O(e^{1/\varepsilon})$ appear in bounds. $\varepsilon \to 0$
- Problem of missing the blowup!

Dominant approach in the literature: rescaling/use of PDE ‘similarity’ properties to ($r$-)adapt/rescale discretisation parameters. Nakagawa ('75), Berger & Kohn ('89), Stuart & Floater ('90), Tourigny & Sanz-Serna ('92), Budd, Huang & Russell ('96) ...

Approach

Construction of adaptive algorithms via rigorous a posteriori error estimates

- Limited literature on a posteriori error control & adaptivity in this context. Karakashian & Plexousakis ('96), Kyza & Makridakis ('11)
- Conditional a posteriori error estimates: final estimates hold under some computationally verifiable conditions
Step back to ODEs...

ODE initial value problem: find $u : [0, T] \to \mathbb{R}$ such that

\[
\frac{du}{dt} = f(u) := u^p, \quad \text{in (0, } T],
\]

$u(0) = u_0,$

with $\mathbb{N} \ni p \geq 2,$ so that the solution blows up in finite time, say $T^*.$

Three different one step schemes: set $U^0 := u_0;$ for $k = 1, \ldots, N,$ solve for $U^k:$

\[
\frac{U^k - U^{k-1}}{\tau_k} = F(U^{k-1}, U^k),
\]

with $F$ one of the following three classical approximations of $f:$

- Explicit Euler \[ F(U^{k-1}, U^k) = f(U^{k-1}), \]
- Implicit Euler \[ F(U^{k-1}, U^k) = f(U^k), \]
- RK2/Improved Euler \[ F(U^{k-1}, U^k) = \frac{1}{2} \left( f(U^{k-1}) + f \left( U^{k-1} + \tau_k f(U^{k-1}) \right) \right). \]
An a posteriori error estimate

Let \( U : [0, T] \to \mathbb{R} \) p/w linear interpolant of \( \{U^k\} \) at \( t^k \), viz

\[
U(t) := \ell_{k-1}(t)U^{k-1} + \ell_k(t)U^k, \quad t \in (t^{k-1}, t^k],
\]

Hence, on each interval \((t^{k-1}, t^k]\), we have

\[
\frac{dU}{dt} = F(U^{k-1}, U^k).
\]

Therefore, on each time interval \((t^{k-1}, t^k]\), the error \( e := u - U \) satisfies

\[
\frac{de}{dt} = f(u) - F(U^{k-1}, U^k) = f(U) + f'(u)e + \sum_{j=2}^{p} \frac{f^{(j)}(U)}{j!}e^j - F(U^{k-1}, U^k),
\]

or, setting \( \eta_k := f(U) - F(U^{k-1}, U^k) \), we have

\[
\frac{de}{dt} = \eta_k + \left( f'(U) + \sum_{j=2}^{p} \frac{f^{(j)}(U)}{j!}e^{j-1} \right)e.
\]
An a posteriori error estimate

Gronwall’s inequality, therefore, implies

$$|e(t)| \leq H_k(t) G_k \phi_k,$$

with $H_k(t) := \exp \left( \sum_{j=2}^{p} \int_{t^{k-1}}^{t} \frac{|f^{(j)}(U)|}{j!} |e|^{j-1} \, ds \right)$, $G_k := \exp \left( \int_{t^{k-1}}^{t} |f'(U)| \, ds \right)$, and $\phi_k := |e(t^{k-1})| + \int_{t^{k-1}}^{t} |\eta_k| \, ds$.

Theorem (Conditional error estimate)

For $k = 1, \ldots, N$, the following a posteriori estimate holds:

$$\max_{t \in [t^{k-1}, t^k]} |e(t)| \leq \delta_k G_k \phi_k,$$

provided that $\delta_k > 1$ is chosen so that

$$\sum_{j=2}^{p} (\delta_k G_k \phi_k)^{j-1} \int_{t^{k-1}}^{t} \frac{|f^{(j)}(U(s))|}{j!} \, ds - \log(\delta_k) = 0.$$
Two algorithms: 1

Algorithm 1 ODE Algorithm 1

1: Input: $f, F, u_0, \tau_1, \text{tol}$.  
2: Compute $U^1$ from $U^0$.  
3: while $\int_{t_0}^{t_1} |\eta_1| \, ds > \text{tol}$ do  
4: $\tau_1 \leftarrow \tau_1/2$.  
5: Compute $U^1$ from $U^0$.  
6: end while  
7: Compute $\delta_1$.  
8: Set $k = 0$.  
9: while $\delta_{k+1}$ exists do  
10: $k \leftarrow k + 1$.  
11: $\tau_{k+1} = \tau_k$.  
12: Compute $U^{k+1}$ from $U^k$.  
13: while $\int_{t_k}^{t_{k+1}} |\eta_{k+1}| \, ds > \text{tol}$ do  
14: $\tau_{k+1} \leftarrow \tau_{k+1}/2$.  
15: Compute $U^{k+1}$ from $U^k$.  
16: end while  
17: Compute $\delta_{k+1}$.  
18: end while  
19: Output: $k, t^k$.  

$\eta_k := f(U) - F(U^{k-1}, U^k)$

Absolute tolerance: $\text{tol}$

$|T_{\text{blowup}} - T_{\text{final}}| \sim N^{-r}$

<table>
<thead>
<tr>
<th>Method</th>
<th>$p = 2$</th>
<th>$p = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implicit Euler</td>
<td>$r \approx 0.66$</td>
<td>$r \approx 0.79$</td>
</tr>
<tr>
<td>Explicit Euler</td>
<td>$r \approx 1.35$</td>
<td>$r \approx 1.60$</td>
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<tr>
<td>Improved Euler</td>
<td>$r \approx 1.2$</td>
<td>$r \approx 1.48$</td>
</tr>
</tbody>
</table>
Two algorithms: 2

Algorithm 2 ODE Algorithm 2

1: Input: $f, F, u_0, \tau_1, \text{tol}$.  
2: Compute $U^1$ from $U^0$.  
3: while $\int_{t_0}^{t_1} |\eta_1| \, ds > \text{tol}$ do  
4: $\tau_1 \leftarrow \tau_1 / 2$.  
5: Compute $U^1$ from $U^0$.  
6: end while  
7: Compute $\delta_1$.  
8: tol = $G_1 \times \text{tol}$.  
9: Set $k = 0$.  
10: while $\delta_{k+1}$ exists do  
11: $k \leftarrow k + 1$.  
12: $\tau_{k+1} = \tau_k$.  
13: Compute $U^{k+1}$ from $U^k$.  
14: while $\int_{t_k}^{t_{k+1}} |\eta_{k+1}| \, ds > \text{tol}$ do  
15: $\tau_{k+1} \leftarrow \tau_{k+1} / 2$.  
16: Compute $U^{k+1}$ from $U^k$.  
17: end while  
18: Compute $\delta_{k+1}$.  
19: tol = $G_{k+1} \times \text{tol}$.  
20: end while  
21: Output: $k, t^k$.  

Relative tolerance: $G_{k+1} \times \text{tol}$  

$|T_{\text{blowup}} - T_{\text{final}}| \sim N^{-r}$

<table>
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<th>$p = 2$</th>
<th>$p = 3$</th>
</tr>
</thead>
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<tr>
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<td>$r \approx 1.00$</td>
<td>$r \approx 1.00$</td>
</tr>
<tr>
<td>Explicit Euler</td>
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<td>$r \approx 1.43$</td>
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<tr>
<td>Improved Euler</td>
<td>$r \approx 2.03$</td>
<td>$r \approx 2.03$</td>
</tr>
</tbody>
</table>
Back to PDEs: time semi-discretisation

\[ 0 := t_0 < t_1 < \cdots < t_N =: T \] partition of \([0, T]\), \(\tau_k := t_{k+1} - t_k\),

**Implicit–Explicit (IMEX) Euler method:** find \(U^k \in H_0^1(\Omega)\), \(k = 0, 1, \ldots, N - 1\):

\[
\frac{U^{k+1} - U^k}{\tau_k} - \Delta U^{k+1} = f(U^k), \text{ with } U^0 = u_0
\]

Why IMEX

- implicit on diffusion \(\Rightarrow\) stability

- explicit on nonlinear reaction \(\Rightarrow\) advantageous approximation near blowup
Error equation

Let \( U : [0, T] \rightarrow H^1_0(\Omega) \) linear interpolant of \( \{U^k\}_k \).

Let \( e := u - U \). Then, for \( f(u) = u^2 \), we have

\[
\partial_t e - \Delta e = 2Ue + e^2 + r_{k+1}
\]

in \( (t_k, t_{k+1}] \) with \( r_{k+1} := [f(U) - f(U^k)] + (t_{k+1} - t)(U^{k+1} - U^k)/\tau_k \).

Energy estimate:

\[
\frac{d}{dt} \|e(t)\|^2 + \|\nabla e(t)\|^2 \leq 4\|U(t)\|_{L^\infty} \|e(t)\|^2 + 2\langle e^2, e \rangle + \|r_{k+1}(t)\|_{-1}^2
\]

Using \( \langle e^2, e \rangle \leq \|e\|_{L^\infty} \|e\|^2 \), Gronwall’s inequality gives

\[
\max_{0 \leq t \leq T} \|e(t)\|^2 \leq \exp\left(2 \int_0^T [2\|U(t)\|_{L^\infty} + \|e(t)\|_{L^\infty}] \, dt\right) \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \|r_{k+1}(t)\|_{-1}^2 \, dt
\]

- Is a fully a posteriori bound possible from this?
- Behaviour of the constant in the run up to blowup?
Exponent growth \textit{(Kyza & Makridakis ('11))}

Gronwall exponent:

\[ \int_0^T 2\|U(t)\|_{L^\infty} + \|e(t)\|_{L^\infty} \, dt \]

On run up to blowup time: \[ \|u(t)\|_{L^\infty} \sim \frac{1}{(T_{\text{blowup}} - t)} \] (Merle & Zaag ('00))

We can infer that we also have, approximately,

\[ \|U(t)\|_{L^\infty}, \|e(t)\|_{L^\infty} \sim \frac{1}{(T_{\text{blowup}} - t)} \]

Hence, at \( T := T_{\text{blowup}} - \varepsilon, \varepsilon > 0, \) the exponent scales like

\[ \int_0^{T_{\text{blowup}} - \varepsilon} \frac{1}{T_{\text{blowup}} - t} \, dt = \ln\left( \frac{T_{\text{blowup}}}{\varepsilon} \right) \]

and, thus, the Gronwall constant scales like

\[ \exp\left( \int_0^{T_{\text{blowup}} - \varepsilon} \frac{1}{T_{\text{blowup}} - t} \, dt \right) \sim \frac{C(T_{\text{blowup}})}{\varepsilon^q} \]

i.e., polynomial growth w.r.t \( \varepsilon \)!  

\textbf{Challenge:} Leads to practical algorithm?
Time-discrete scheme a posteriori bound of \( (Kyza \ & \ Makridakis \ (’11)) \)

fixed point arguments + semigroup theory \( \implies \) a posteriori estimate

### Conditional a posteriori error estimates

- **\( L^\infty(L^\infty) \):** \[ \| e \|_{L^\infty(L^\infty)} \leq e^{1/8 + 4 \int_0^T \| U(s) \|_{L^\infty} \, ds} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \| r_{k+1}(s) \|_{L^\infty} \, ds \]

- **\( L^\infty(L^2) \):** \[ \max_{0 \leq t \leq T} \| e(t) \|^2 \leq e^{1/8 + 4 \int_0^T \| U(s) \|_{L^\infty} \, ds} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \| r_{k+1}(s) \|^2_{-1} \, ds \]

provided \( \tau_k \) are chosen so that

\[ e^4 \int_0^T \| U(s) \|_{L^\infty} \, ds \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \| r_{k+1}(s) \|_{L^\infty} \, ds \leq \frac{3}{16} \rho, \quad \text{with } \rho < \frac{1}{16 T} \]

---

**Global condition! efficient time-adaptive algorithm?** – More local conditions?

- For fully-discrete \( L^\infty \)-norm a posteriori bounds for resp. elliptic problem are needed in this framework.

- growth range \( u^p, \ p > 1 \).
A new a posteriori bound (fully discrete bound also available)

New conditional estimate

\[
\max_{t_k \leq t \leq t_{k+1}} \| e(t) \|^2 \leq \delta_{k+1} e^4 \int_{t_k}^{t_{k+1}} \| U(s) \|_{L^\infty} \, ds \left[ \| e(t_k) \|^2 + \int_{t_k}^{t_{k+1}} \| r_{k+1}(s) \|_{-1}^2 \, ds \right]
\]

with \( e(0) = 0 \), where \( \delta_{k+1} > 1 \) is chosen to satisfy

\[
\beta^2 \delta_{k+1} e^4 \int_{t_k}^{t_{k+1}} \| U(s) \|_{L^\infty} \, ds \left[ \| e(t_k) \|^2 + \int_{t_k}^{t_{k+1}} \| r_{k+1}(s) \|_{-1}^2 \, ds \right] \tau_k - \ln \delta_{k+1} = 0
\]

Ingredients:

- Gagliardo–Nirenberg inequality: \( \langle e^2, e \rangle(t) \leq \beta \| e(t) \|^2 \| \nabla e(t) \| \)

- a new local on each time-step continuation argument

("global" continuation arguments in this spirit: Kessler, Nochetto & Schmidt (’04), Bartels (’05), Cangiani, G. & Jensen (’13))
Condition – discussion

Condition satisfied only if

\[ \beta^2 e^4 \int_{t_k}^{t_{k+1}} \| U(s) \|_{L^\infty} \, ds \left[ \| e(t_k) \|^2 + \int_{t_k}^{t_{k+1}} \| r_{k+1}(s) \|_{-1}^2 \, ds \right] \tau_k < \frac{1}{e} \]

This implies restriction on time-steps \( \tau_k \), i.e., conditional estimates.

Is condition practical?

- Say \( t_{k+1} = T_{\text{blowup}} - \varepsilon \), then \( e^4 \int_{t_k}^{t_{k+1}} \| U(s) \|_{L^\infty} \, ds \sim \left( \frac{\tau_k + \varepsilon}{\varepsilon} \right)^q \)

- \( \int_{t_n}^{t_{n+1}} \| r_{n+1}(s) \|_{-1}^2 \, ds \) estimable via a posteriori bounds

\( \delta_{n+1} \) computed via, e.g., Newton’s method.
Semilinear advection-diffusion problem with blowup

\[
\begin{align*}
    u_t - \epsilon \Delta u + \mathbf{a} \cdot \nabla u &= u^2 + g & \text{in } \Omega \times (0, T], \\
    u &= 0 & \text{on } \partial \Omega \times (0, T], \\
    u(\cdot, 0) &= u_0 & \text{in } \Omega,
\end{align*}
\]

Spatial discretisation: Discontinuous Galerkin (dG) method with upwind flux.

**dG IMEX method:** find \( U_h^{k+1} \in \mathbb{V}_h^{k+1} \) s.t.

\[
\left\langle \frac{U_h^{k+1} - U_h^k}{\tau_k}, V_h \right\rangle + B(t_{k+1}; U_h^{k+1}, V_h) = \left\langle f(U_h^k) + g^k, V_h \right\rangle \quad \forall V_h \in \mathbb{V}_h^{k+1}
\]

Continuation Argument + Elliptic Reconstruction

Conditional a posteriori error estimate in the \( L^\infty(L^2) \)-norm for dG IMEX

For elliptic reconstruction see Makridakis & Nochetto ('03), Lakkis & Makridakis ('06), G. Lakkis & Virtanen ('11), Cangiani, G., & Metcalfe ('14)

Key attribute of the approach: Flexibility on the elliptic operator!
A simple test case: reaction-diffusion problem

\[ \Omega = (-4, 4) \times (-4, 4), \; u_0(x, y) = 10e^{-2(x^2+y^2)} \rightsquigarrow \text{single point blowup} \]

- space-time adaptive algorithm
- when condition fails, restarts with smaller timestep
A simple test case

<table>
<thead>
<tr>
<th>tol</th>
<th># time-steps</th>
<th>( T )</th>
<th>( | U_h(T) |_{L^\infty} )</th>
<th>Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0.125)^{10} )</td>
<td>6956</td>
<td>0.21228</td>
<td>238.705</td>
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<td>( (0.125)^{11} )</td>
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<td>( (0.125)^{12} )</td>
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<td>( (0.125)^{13} )</td>
<td>35580</td>
<td>0.21549</td>
<td>722.884</td>
<td>157300.0</td>
</tr>
</tbody>
</table>

\[
\| U_h(t) \|_{L^\infty} \sim \frac{1}{(T^* - t)^{p_k}}
\]

Plots depict: \( p_k \) vs. \( \frac{1}{T^* - t} \)
Observed rates of convergence

- $\| U_h(t) \|_{L^\infty}$ appears to blow up at the expected rate on the run-up to $T^*$
- we have $|T^* - T| \sim N^{-1/2}$
  - shortcoming of energy method?
  - semigroup techniques?
- Estimator blows up at faster rate, but delivers optimal blowup rate for the numerical solution regardless!
Numerical experiment – advection-diffusion problem

\( \Omega = (-4, 4) \times (-4, 4), \kappa = 1, a \equiv (1, 1)^T, g \equiv 1, u_0 \equiv 0 \)

<table>
<thead>
<tr>
<th>tol</th>
<th># time-steps</th>
<th>( T )</th>
<th>( | U_h(T) |_{L^\infty} )</th>
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<tr>
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<tr>
<td>((0.125)^3)</td>
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<td>((0.125)^5)</td>
<td>520</td>
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<td>((0.125)^6)</td>
<td>1064</td>
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<td>71352</td>
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</table>
Numerical experiment – blowup on 1D manifold

\( \Omega = (-8, 8)^2, \ \kappa = 1, \ \mathbf{a} = (0, 0)^T, \ f_0 = 0 \)

‘volcano’ type initial condition be given by \( u_0 = 10(x^2 + y^2) \exp(- (x^2 + y^2)/2) \)
Numerical experiment – blowup on 1D manifold

$\Omega = (-8, 8)^2$, $\kappa = 1$, $a = (0, 0)^T$, $f_0 = 0$

‘volcano’ type initial condition be given by $u_0 = 10(x^2 + y^2) \exp(-(x^2 + y^2)/2)$

<table>
<thead>
<tr>
<th>$t_{tol}^+$</th>
<th>Time Steps</th>
<th>Estimator</th>
<th>Final Time</th>
<th>$|U_h(T)|_{L^\infty(\Omega)}$</th>
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<tbody>
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<td>10</td>
<td>63</td>
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<td>$(0.125)^9$</td>
<td>16137</td>
<td>287420</td>
<td>0.16442</td>
<td>505.236</td>
</tr>
<tr>
<td>$(0.125)^{10}$</td>
<td>32846</td>
<td>331848</td>
<td>0.16512</td>
<td>769.652</td>
</tr>
<tr>
<td>$(0.125)^{11}$</td>
<td>66442</td>
<td>626522</td>
<td>0.16558</td>
<td>1175.21</td>
</tr>
</tbody>
</table>
Notation

Consider an open polygonal domain $\Omega \subset \mathbb{R}^d$ subdivided into two subdomains $\Omega_1$ and $\Omega_2$:

\[ \Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_i \]
\[ \Gamma_i = \bar{\Omega}_1 \cap \bar{\Omega}_2 \]

and set $H^1 := [H^1(\Omega_1 \cup \Omega_2)]^n$, $n \in \mathbb{N}$.

PDE system:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u - UB) + F(u) &= 0 \quad \text{in } (0, T] \times (\Omega_1 \cup \Omega_2), \\
\quad u(0, x) &= u_0(x) \quad \text{on } \{0\} \times \Omega, \\
\quad u &= g_D \quad \text{on } \Gamma_D, \\
(A \nabla u - \chi^- U B) n &= g_N \quad \text{on } \Gamma_N,
\end{align*}
\]

where $\chi^-$ is the (vector-valued) characteristic function of the inflow part of $\partial \Omega$ and $U := \text{diag}(u)$. On $\Gamma_i$ we impose:

\[
\begin{align*}
(A \nabla u - UB)n|_{\Omega^1} &= P(u)(u^2 - u^1) - \{U\}_w RB n^1 \\
(A \nabla u - UB)n|_{\Omega^2} &= P(u)(u^1 - u^2) - \{U\}_w RB n^2
\end{align*}
\]
Reflective membranes – time-dependent sharp features

- a DG method & a priori analysis with solution boundedness assumption

- No analytical information – numerics?

Cangiani, G., Jensen ('13)
A posteriori error bounds for dG on curved geometries

\[ \Omega \subset \mathbb{R}^d, \ d = 2, 3. \ \Omega = \Omega_1 \cup \Omega_2 \cup \Gamma^{tr}, \text{ with } \Gamma^{tr} := (\partial \Omega_1 \cap \partial \Omega_2) \setminus \partial \Omega \text{ Lipschitz.} \]

\[ -\Delta u = f, \quad \text{in } \Omega_1 \cup \Omega_2, \]
\[ u = 0, \quad \text{on } \partial \Omega, \]
\[ \mathbf{n}^1 \cdot \nabla u_1 = C_{tr}(u_2 - u_1)|_{\Omega_1} \quad \text{on } \bar{\Omega}_1 \cap \Gamma^{tr}, \]
\[ \mathbf{n}^2 \cdot \nabla u_2 = C_{tr}(u_1 - u_2)|_{\Omega_2} \quad \text{on } \bar{\Omega}_2 \cap \Gamma^{tr}, \]

where \( u_i = u|_{\bar{\Omega}_i \cap \Gamma^{tr}}, \ i = 1, 2, \ C_{tr} \) a given permeability constant.
Conclusions

- ‘Energy methods’ for error control are relevant and competitive for nonlinear evolution problems
- Error control for curved interfaces/boundaries
Some references


Computation of numerical blowup time and blowup rate

Let $t^*$ denote a numerical blowup time. We implement a set numerical experiments (corresponding to different tolerances) producing $U_h^\ell$, $\ell = 1, \cdots, L$ approximations to the exact solution $u$. Assume that

$$\|U_h^\ell(t_n)\|_{L^\infty} \sim \left(\frac{1}{t^* - t_n}\right)^p$$

1. Since $f(u) = u^2$, assume that $p = 1$, and use $\|U_h^L(t_{N-1})\|_{L^\infty}, \|U_h^L(T)\|_{L^\infty}$ to calculate

$$\begin{align*}
\|U_h^L(t_{N-1})\|_{L^\infty} &= C_L \frac{1}{t^* - t_{N-1}} \\
\|U_h^L(T)\|_{L^\infty} &= C_L \frac{1}{t^* - T}
\end{align*}$$

$$\Rightarrow \quad t^* = \frac{T\|U_h^L(T)\|_{L^\infty} - t_{N-1}\|U_h^L(t_{N-1})\|_{L^\infty}}{\|U_h^L(t_{N-1})\|_{L^\infty} - \|U_h^L(t_{N-1})\|_{L^\infty}}$$

• For the considered example, $t^* = 0.21705$

2. Consider $t^*(= 0.21705)$ as the numerical blowup time. We use $\|U_h^\ell(t)\|_{L^\infty}$, $\ell \neq L$ to compute the numerical blowup time:

$$p_n := \frac{\ln \left(\|U_h^\ell(t_{n+1})\|_{L^\infty} / \|U_h^\ell(t_n)\|_{L^\infty}\right)}{\ln \left((t^* - t_n)/(t^* - t_{n+1})\right)}$$

• We expect $p_n \to 1$ as $n \to N$, for the considered model problem.