Adaptive mesh redistribution on the sphere for global atmospheric modelling

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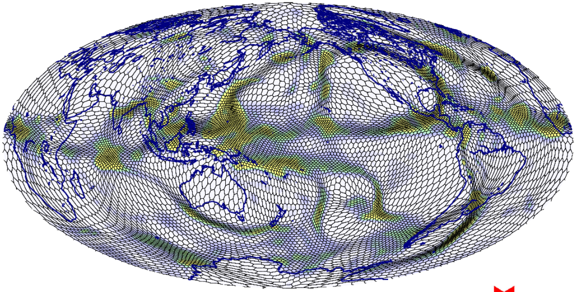


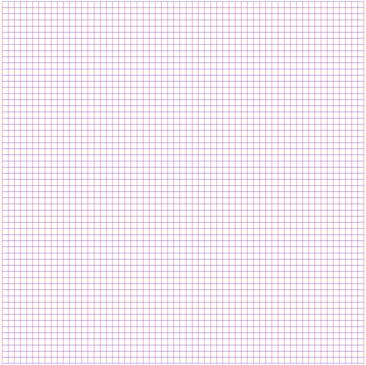
NATURAL ENVIRONMENT RESEARCH COUNCIL

Why r-adaptivity

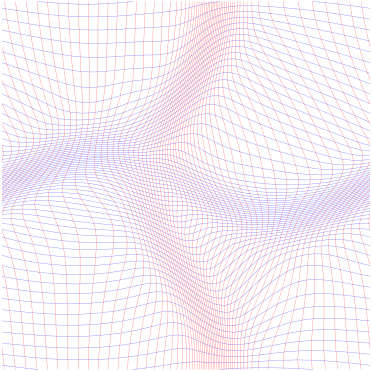
- Does not create load balancing problems on parallel computers,
- Does not require mapping solutions between different meshes,
- Does not lead to sudden changes in resolution,
- Can be retro-fitted into existing models

Objectives of the NERC project

- Solve optimal transport equations on the sphere to efficiently redistribute a mesh
- Assess mesh quality for the equations of the atmosphere
- Develop mimetic finite element/volume methods on moving meshes
- Compare with established test cases
- Establish suitable refinement criteria for the atmosphere



Original computational mesh $\mathcal{T}_c$



Adapted physical mesh $\mathcal{T}_p$

$$F(\mathcal{T}_c) = \mathcal{T}_p; \qquad \forall \xi \in \mathcal{T}_c \; \exists x \in \mathcal{T}_p \text{ s.t. } x = F(\xi) \tag{1}$$

Monitor function equidistribution

Given $m(x) > 0$, find $F : \Omega_c \to \Omega_p$ such that

$$m(x)|J(\xi)| = c. \tag{2}$$

Optimally transported meshes

Seek F^* such that

$$F^* = \arg\min_F ||F - I|| = \int_{\Omega_c} |\xi - F(\xi)|^2 \, \mathrm{d}\xi. \tag{3}$$

Optimally transported meshes

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$$F^* = \arg\min_F ||F - I|| = \int_{\Omega_c} |\xi - F(\xi)|^2 \, \mathrm{d}\xi. \qquad (3)$$

Theorem (Brenier (1991) [in cuboid domains])

There exists a **unique** *optimally transported map $\mathbf{F}(\xi)$ which minimises (3), and the Jacobian of which satisfies the equidistribution equation (2). Furthermore, $\mathbf{F}(\xi)$ can be written as the gradient (with respect to ξ) of a convex scalar (mesh) potential $\phi(\xi)$, so that*

$$x(\xi) = \nabla_\xi \phi(\xi), \qquad H_\xi(\phi(\xi)) \succ 0. \qquad (4)$$

Brenier, Y. (1991). Polar Factorization and Monotone Rearrangement of Vector-Valued Functions.

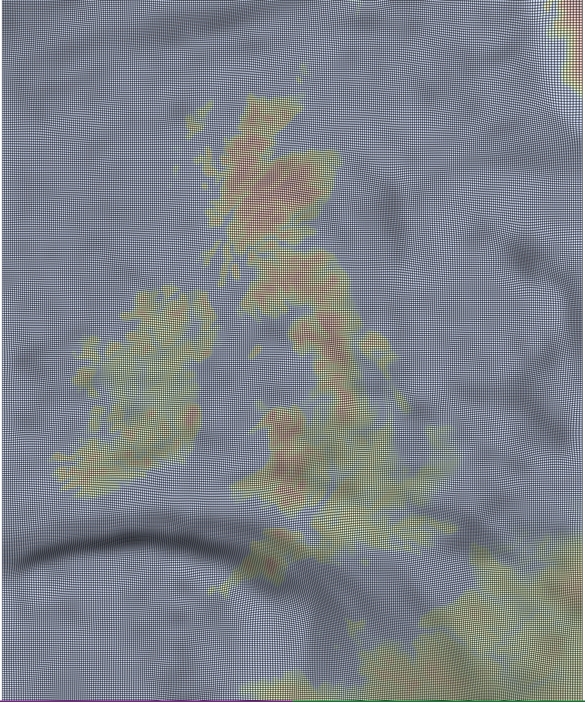
Communications on Pure and Applied Mathematics, XLIV:375–417

A Monge-Ampère equation

As $x = \nabla\phi$ and $m(x)|J(\xi)| = c$,

$$m(\nabla\phi)|H(\phi)| = c \quad \Longleftrightarrow \quad |H(\phi)| = \frac{c}{m(\nabla\phi)} \tag{5}$$

$$m(\nabla\phi)\frac{V_i(x)}{V_i(\xi)} = c \quad \Longleftrightarrow \quad \frac{V_i(x)}{V_i(\xi)} = \frac{c}{m(\nabla\phi)} \tag{6}$$



Theorem (McCann (2001))

Let M be a connected, complete smooth Riemannian manifold, equipped with its standard volume measure dx. Let μ, ν be two probability measures on M with compact support, and let the objective function $c(\xi, x)$ be equal to $d(\xi, \mathbf{x})^2$, where d is the geodesic distance on M. Further, assume that μ is absolutely continuous with respect to the volume measure on M. Then, there is a unique optimal transport map F where F pushes forward the measure μ onto ν. Then, (using classical optimal transport notation):

$$F_{\#}(\mu) = \nu \quad \textit{i.e.} \quad \mathbf{x} = F(\xi) = \exp_{\xi}[\nabla\phi(\xi)] \tag{7}$$

for some $d^2/2$-convex potential ϕ.

McCann, R. (2001). Polar factorization of maps on Riemannian manifolds.

Geometric & Functional Analysis GAFA, 11(3):589–608

Corollary (Weller, B., Budd, Cullen (2015))

There exists a unique, optimally transported mesh on the sphere that satisfies the equidistribution principle. Moreover, that mesh is defined by a c-convex scalar potential function that satisfies the Monge-Ampère type equation

$$m(\exp_\xi[\nabla\phi(\xi)])|J(\xi)| = c. \tag{8}$$

Corollary (Weller, B., Budd, Cullen (2015))

The optimally transported mesh on the sphere satisfying the equidistribution principle does not exhibit tangling.

Weller, H., Browne, P., Budd, C., and Cullen, M. (2015). Mesh adaptation on the sphere using optimal transport and the numerical solution of a Monge-Ampère type equation.

J Comp Phys, (In Press)

Solution techniques for the Monge-Ampère equation

Parabolic Relaxation, Budd & Williams (2009)

$$(I - \gamma\nabla^2)\phi^{n+1} = (I - \gamma\nabla^2)\phi^n + \delta t \left[m(x^n)|I + H(\phi^n)|\right]^{\frac{1}{d}}. \quad (9)$$

Linearisation about 0

$$|I + H(\phi^{n+1})| = 1 + \nabla^2\phi^{n+1} + \mathcal{N}(\phi^{n+1}) \quad (10)$$

Linearisation about ϕ^n

$$|I + H(\phi^{n+1})| = |I + H(\phi^n)| + \varepsilon\nabla \cdot A^n\nabla\psi + \mathcal{N}(\varepsilon\psi) \quad (11)$$

Budd, C. and Williams, J. (2009). Moving mesh generation using the parabolic Monge-Ampère Equation.

SIAM Journal on Scientific Computing, 31(5):3438–3465

Solution techniques for the Monge-Ampère equation

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Fixed point iterations, Weller, B., Budd, Cullen (2015)

$$\gamma \nabla^2 \phi^{n+1} = \gamma \nabla^2 \phi^n - |I + H(\phi^n)| + \frac{c}{m(x^n)}, \qquad \forall n \in \mathbb{N}. \quad (10)$$

Adaptive linearisation fixed point iterations

$$\nabla \cdot \left(A^n \nabla \phi^{n+1} \right) = \nabla \cdot (A^n \nabla \phi^n) - |I + H(\phi^n)| + \frac{c}{m(x^n)}, \qquad \forall n \in \mathbb{N}. \quad (11)$$

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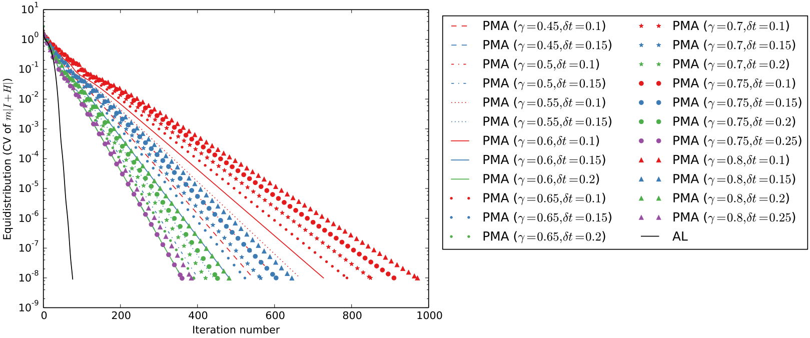
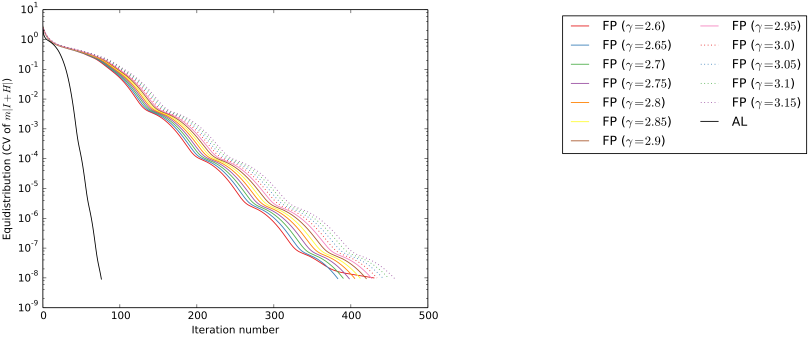
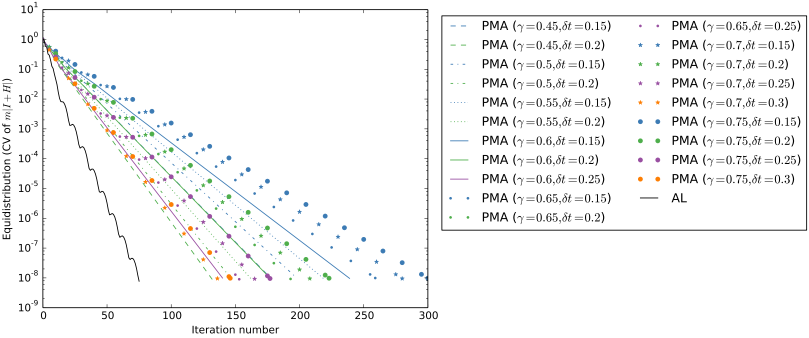
Convergence 1



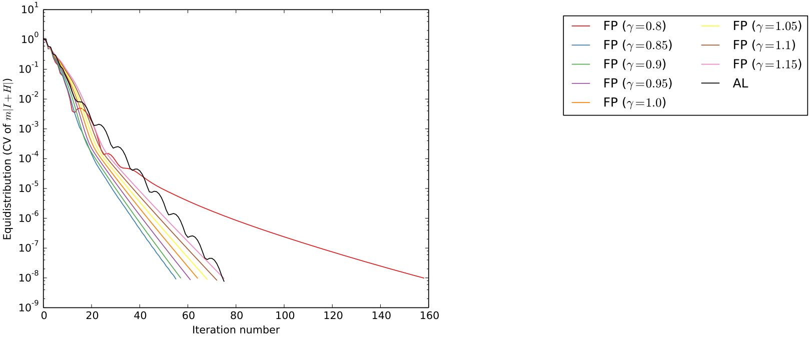
Convergence 2



Convergence 3



Convergence 4

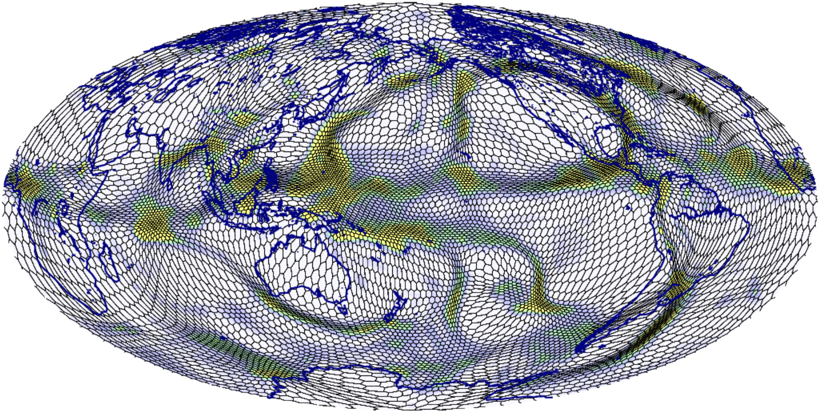


Mesh redistribution on the sphere

$$\frac{V_i(x)}{V_i(\xi)} = \frac{c}{m(x)} \qquad \text{where} \qquad x = \exp_\xi(\nabla\phi) \qquad \text{on } \mathcal{S}^2 \qquad (12)$$

- Finite volume method (OpenFOAM)
- Fixed point iterations
- Geometric version of the Hessian
- Linearisation about 0 on a tangent plane
- Exponential mappings of the points
- Monitor function derived from reanalysis precipitation data
- Hexagonal isocohedral $\mathcal{T}_c$

Mesh redistribution on the sphere



Thank you for listening