Stabilised finite element methods in anisotropic quadrilateral meshes

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1 Motivation: Some general considerations on inf-sup conditions.
2 The Stokes problem.
3 The $Q_{k+1}^2 \times P_{k-1}$ pair.
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4 The $Q_1^2 \times P_0$ pair.
5 The Oseen equation.
6 Conclusions, open questions.
Motivation: A saddle point problem

The setting: Find \((u, p) \in V \times Q\) such that

\[
\begin{align*}
    a(u, v) + b(v, p) &= F(v) \quad \forall v \in V, \\
    b(u, q) &= 0 \quad \forall q \in Q.
\end{align*}
\]

The Galerkin scheme: Given \(V_h \subset V\) and \(Q_h \subset Q\), finite-dimensional spaces: Find \((u_h, p_h) \in V_h \times Q_h\) such that

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\begin{align*}
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\]

Remark: The stability and error estimates constants vary as \(\beta^{-2}\), where \(\beta\) is the discrete inf-sup constant.
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Remark: The stability and error estimates constants vary as \(\beta^{-2}\), where \(\beta\) is the discrete inf-sup constant.
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Problem: In many interesting cases, $\beta$ degenerates with some important quantity (e.g., the aspect ratio). Fortunately, in some cases, the following decomposition can be proved:

$$Q_h = Q_h^G \oplus Q_h^B,$$

where $V_h \times Q_h^G$ satisfies:

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{b(v_h, q_h)}{\|v_h\|_V} \geq \beta_G \|q_h\|_Q \quad \forall q_h \in Q_h^G,$$

where $\beta_G > 0$ does not depend on any bad parameter.

Then, the following weak inf-sup condition can be proved:

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{b(v_h, q_h)}{\|v_h\|_V} \geq \beta_G \|q_h\|_Q - C\|q_h - \Pi q_h\|_Q \quad \forall q_h \in Q_h,$$

where $\Pi : Q \to Q_h^G$ is any continuous linear projection onto the good space $Q_h^G$, and $C > 0$ is an $O(1)$ constant.
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Two possible solutions:

• If $Q_h^G$ is essentially equal to $Q_h$, just use the pair $V_h \times Q_h^G$ as a mixed method; else

• A stabilised variant, in the vein of the minimal stabilisation (cf. Brezzi & Fortin): Find $(u_h, p_h) \in V_h \times Q_h$ such that

\[
\begin{align*}
    a(u_h, v_h) + b(v_h, p_h) &= F(v_h) \quad \forall v_h \in V_h, \\
    -b(u_h, q_h) + ((I - \Pi)p_h, (I - \Pi)q_h)_Q &= 0 \quad \forall q_h \in Q_h.
\end{align*}
\]

For both variants, stability and convergence, with constants depending only on $\beta_G$, can be proved.
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Two possible solutions:

- If $Q^G_h$ is essentially equal to $Q_h$, just use the pair $V_h \times Q^G_h$ as a mixed method; else
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For both variants, stability and convergence, with constants depending only on $\beta_G$, can be proved.
The Stokes problem

The Stokes problem: Find a pair \((u, p)\) such that \(u = 0\) on \(\partial \Omega\), and

\[
-\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \Omega.
\]

The finite element spaces: For a given partition \(\mathcal{P}\) and an integer \(k \geq 1\) we define:

\[
V_{\mathcal{P}} = \{ v \in H^1_0(\Omega)^2 : v \circ F_K \in Q_{k+1}^2 \quad \forall K \in \mathcal{P} \}
\]

\[
\mathcal{M}_{\mathcal{P}} = \{ q \in L^2_0(\Omega) : q \circ F_K \in P_{k-1} \quad \forall K \in \mathcal{P} \}
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Remark: This pair of spaces is inf-sup stable on regular meshes, and in anisotropic edge patches.
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V_{\mathcal{P}} &= \{ v \in H_0^1(\Omega)^2 : v \circ F_K \in \mathbb{Q}_{k+1}^2 \quad \forall K \in \mathcal{P} \} \\
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Figure 1: Typical anisotropically refined corner patches $\Omega_c$ with the corresponding subsets $\omega_c$ shown shaded. On the left: a single corner patch.
The Stokes problem

Figure 2: A typical example of an anisotropically refined mesh.
Lemma (A & C 2000)

There exists $C > 0$ such that

$$\inf_{q \in \mathcal{M}_p} \sup_{v \in \mathcal{V}_p} \frac{(\nabla \cdot v, q)_\Omega}{|v|_{1,\Omega} \|q\|_{0,\Omega}} = \beta_p = C k^{-1/2} \min\{1, k \sqrt{\varrho}\},$$

where $\varrho = h_c/H_c$ is the mesh aspect ratio.
The Stokes problem

Figure 3: Behaviour of the inf-sup constants $\beta_P$ and $\tilde{\beta}_P$ with respect to the aspect ratio and polynomial degree $k = 4$ on the T-mesh.
Whose fault is that?

\[ \beta_{\mathcal{P}_a} = C \]

\[ \beta_{\mathcal{P}_b} = Ck^{-1/2} \]

Together, ...
Whose fault is that?

\[ \beta_{\mathcal{P}_a} = C \]
\[ \beta_{\mathcal{P}_b} = C k^{-1/2} \]

Together, \( \mathcal{M}_P = \mathcal{M}_{\mathcal{P}_a} \oplus \mathcal{M}_{\mathcal{P}_b} \oplus \text{Span}\{q_B^c\} \) where

\[ q_B^c = \begin{cases} 
1 & \text{in } \omega_c, \\
-\frac{|\omega_c|}{|\Omega_c \setminus \omega_c|} & \text{in } \Omega_c \setminus \omega_c,
\end{cases} \]

and \( \omega_c \) is the shaded, extremely small subdomain.
Whose fault is that?

Theorem (Corollary of A& C 2000)

Let $\mathcal{M}_\mathcal{P}^* = \{ q \in \mathcal{M}_\mathcal{P} : \int_{\omega_c} q = 0, \text{ for all corner patches} \}$. Then, there is a positive constant $C$, independent of any aspect ratio such that:

$$\inf_{q \in \mathcal{M}_\mathcal{P}^*} \sup_{\mathbf{v} \in \mathbf{V}_\mathcal{P}} \frac{(\nabla \cdot \mathbf{v}, q)_{\Omega}}{||\mathbf{v}||_{1,\Omega} ||q||_{0,\Omega}} \geq C k^{-1/2}.$$ 

Consequence 1: The pair $\mathbf{V}_\mathcal{P} \times \mathcal{M}_\mathcal{P}^*$ is a uniformly inf-sup stable pair.

Consequence 2: Confirmation that the culprit of the inf-sup deficiency is only one pressure mode per corner patch. Namely, the function $q^P_B$ defined previously. Then, it is very easy to propose a stabilised finite element method using a minimal stabilisation idea.
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Theorem (Corollary of A& C 2000)

Let \( M_\gamma^* = \{ q \in M_\gamma : \int_{\omega_c} q = 0, \text{ for all corner patches} \} \). Then, there is a positive constant \( C \), independent of any aspect ratio such that:

\[
\inf_{q \in M_\gamma^*} \sup_{v \in V_\gamma} \frac{(\nabla \cdot v, q)_\Omega}{|v|_{1,\Omega} \|q\|_{0,\Omega}} \geq Ck^{-1/2}.
\]

Consequence 1: The pair \( V_\gamma \times M_\gamma^* \) is a uniformly inf-sup stable pair.

Consequence 2: Confirmation that the culprit of the inf-sup deficiency is only one pressure mode per corner patch. Namely, the function \( q^c_B \) defined previously. Then, it is very easy to propose a stabilised finite element method using a minimal stabilisation idea.
First Solution

The first stabilised method: Find \((u_{p}, p_{p}) \in V_{p} \times M_{p}\) such that:

\[
B((u_{p}, p_{p}), (v, q)) - \tilde{s}(p_{p}, q) = (f, v)_{\Omega} \quad \forall (v, q) \in V_{p} \times M_{p},
\]

where

\[
B((u, p), (v, q)) := (\nabla u, \nabla v)_{\Omega} - (p, \nabla \cdot v)_{\Omega} - (q, \nabla \cdot u)_{\Omega},
\]

\[
\tilde{s}(p, q) := ((I - \Pi) p, (I - \Pi) q)_{\Omega}.
\]

- Pros: Stability can be proved quite easily, and numerics follow.
- Cons: The consistency error can not be bounded optimally.
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\]

- Pros: Stability can be proved quite easily, and numerics follow.
- Cons: The consistency error can not be bounded optimally.
Important Remark: There exists a projection operator $\tilde{\Pi}$ such that

$$\tilde{s}(p, q) = ((I - \tilde{\Pi})p, (I - \tilde{\Pi})q)_{\Omega} = \frac{\tau_{\gamma_c}}{k^2} \int_{\gamma_c} [p] \int_{\gamma_c} [q],$$

where $\tau_{\gamma_c} > 0$ is an appropriate constant, and $\gamma_c$ is a single, arbitrary, edge connecting a small square element $\kappa_c$ in $\Omega_c$ with a stretched element $K_c$. 
Reminder of the T-mesh

(a) Edge patch

(b) Corner patch
The new mixed method

The useful consequence:

**Theorem**

Let \( \tilde{M}_P \subset M_P \) denote the subspace defined by

\[
\tilde{M}_P = \{ q \in M_P : \int_{\gamma_c} [q] = 0 \text{ for all corner patches} \}.
\]

Then, the following inf-sup stability holds

\[
\inf_{q \in \tilde{M}_P} \sup_{v \in V_P} \frac{(\nabla \cdot v, q)_\Omega}{|v|_{1,\Omega} \| q \|_{0,\Omega}} \geq \tilde{\beta}_P > 0,
\]

where

\[
\tilde{\beta}_P = \max\{\beta_P, C k^{-3/2}\}.
\]

Moreover, if \( p \in H^1(\Omega) \), then there exists a positive constant \( C \) such that

\[
\inf_{\tilde{q}_P \in \tilde{M}_P} \| p - \tilde{q}_P \|_{0,\Omega}^2 \leq C \inf_{q_P \in M_P} (\| p - q_P \|_{0,\Omega}^2 + \sum_c \frac{|\gamma_c|^2}{k^2} \| \partial_n (p - q_P) \|_{0,\kappa_c \cup K_c}^2).
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$$\tilde{\mathcal{M}}_P = \{ q \in \mathcal{M}_P : \int_{\gamma_c} [q] = 0 \text{ for all corner patches} \} .$$

Then, the following inf-sup stability holds

$$\inf_{q \in \tilde{\mathcal{M}}_P} \sup_{v \in \mathbf{V}_P} \frac{(\nabla \cdot v, q)_\Omega}{|v|_{1,\Omega} \|q\|_{0,\Omega}} \geq \tilde{\beta}_P > 0,$$

where

$$\tilde{\beta}_P = \max\{\beta_P, C k^{-3/2}\} .$$

Moreover, if $p \in H^1(\Omega)$, then there exists a positive constant $C$ such that

$$\inf_{\tilde{q}_P \in \tilde{\mathcal{M}}_P} \|p - \tilde{q}_P\|_{0,\Omega}^2 \leq C \inf_{q_P \in \mathcal{M}_P} (\|p - q_P\|_{0,\Omega}^2 + \sum_c \frac{|\gamma_c|^2}{k^2} \|\partial_n (p - q_P)\|_{0,\kappa_c \cup K_c}^2) .$$
The new mixed method: Some numerics

\[ k = 1 \]

\[ \beta (k = 1) \]

\[ \tilde{\beta} \]

\[ \beta_P \]

\[ \tilde{\beta}_P \]

\[ \log_{10}(\varrho) \]

\[ \log_{10} (\beta) \]

\[ \log_{10} (\tilde{\beta}) \]

\[ \log_{10} (\beta_P) \]

\[ \log_{10} (\tilde{\beta}_P) \]

\[ \beta (k = 1) \]

\[ \tilde{\beta} (k = 1) \]

\[ \beta_P (k = 1) \]

\[ \tilde{\beta}_P (k = 1) \]

\[ \varrho \]

\[ (k = 1) \]
The new mixed method: Some numerics

\[ \tilde{\beta}_P (k = 2) \]

\[ \beta_P \]

\[ \log_{10}(\rho) \]

\[ (k = 2) \]
The new mixed method: Some numerics

\[ \tilde{\beta}_P (k = 6) \]

\[ \beta_P \]
The new mixed method: Some numerics

\[ \tilde{\beta}_P \quad \beta_P \]

\( \varrho \quad (k = 10) \)
The new mixed method: Some numerics

\[ \begin{align*}
10^{-5} & \quad 10^{-4} & \quad 10^{-3} & \quad 10^{-2} & \quad 10^{-1} \\
\bar{\beta} & \quad \beta & \quad (k = 18)
\end{align*} \]
The new mixed method: Some numerics

\[ \kappa (k = 19) \]

\[ \beta, \tilde{\beta} \]

\[ 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1} \]

\[ \varrho (k = 19) \]
The new mixed method: Some numerics

\[ \beta_p \hspace{1cm} \tilde{\beta}_p \]

\( \rho \) \( (k = 20) \)
The stabilised alternative

The stabilised method: Find \((u_P, p_P) \in V_P \times M_P\) such that:

\[
B((u_P, p_P), (v, q)) - S(p_P, q) = (f, v)_\Omega \quad \forall (v, q) \in V_P \times M_P,
\]

where

\[
S(p, q) := \frac{1}{k^2} \sum_c \int_{\gamma_c} [p] \cdot \int_{\gamma_c} [q].
\]

Lemma

There exist positive constants \(C_1, C_2\) such that for all \(q_P \in M_P\),

\[
C_1\|q_P - \tilde{\Pi}_P q_P\|_\Omega^2 \leq S(q_P, q_P) \leq C_2\|q_P - \tilde{\Pi}_P q_P\|_\Omega^2.
\]

Furthermore,

\[
S(q_P, q_P) \leq C \left\{ \|q_P\|_{0, \Omega}^2 + k^{-2} \sum_c (\|p - q_P\|_{0, \kappa_c \cup K_c}^2 + \|\gamma_c\|^2 \|\partial n_c (p - q_P)\|_{\kappa_c \cup K_c}^2) \right\}
\]

for all \(p \in H^1(\Omega)\).
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**Lemma**

There exist positive constants \(C_1, C_2\) such that for all \(q_{p} \in M_{p}\),

\[
C_1 \|q_{p} - \tilde{\Pi}_{p} q_{p}\|_{\Omega}^{2} \leq S(q_{p}, q_{p}) \leq C_2 \|q_{p} - \tilde{\Pi}_{p} q_{p}\|_{\Omega}^{2}.
\]

Furthermore,

\[
S(q_{p}, q_{p}) \leq C \left\{ \|q_{p}\|_{0,\Omega}^{2} \left[ k^{-2} \sum_{c} (\|p - q_{p}\|_{0,\kappa_{c}\cup K_{c}}^{2} + |\gamma_{c}|^{2} \|\partial_{n_{c}} (p - q_{p})\|_{\kappa_{c}\cup K_{c}}^{2}) \right] \right\}
\]

for all \(p \in H^{1}(\Omega)\).
The stabilised alternative

**Theorem**

For all \((w, r) \in V_P \times M_P\), there holds

\[
\sup_{(v, q) \in V_P \times M_P} \frac{B_s((w, r), (v, q))}{\|(v, q)\|} \geq C\bar{\beta}_P^2 \|\!(w, r)\|.
\]

Moreover, if \(p \in H^1(\Omega)\), then there exists a positive constant \(C\) such that

\[
\|(u - u^s_P, p - p^s_P)\| \leq (1 + C\bar{\beta}_P^{-2}) \inf_{(v_P, p_P) \in V_P \times M_P} \left\{ \|(u - v_P, p - q_P)\| + k^{-1} \left( \sum_c |\gamma_c|^2 \|\partial_n c (p - q_P)\|_{0, \kappa_c \cup K_c}^2 \right)^{1/2} \right\}.
\]
The stabilised alternative

Figure 4: Behaviour of inf-sup constants of different methods on the T-mesh shown in Figure 1 for fixed aspect ratio $\varrho = 10^{-4}$ and increasing polynomial degree $k$. 
The $Q_1 \times P_0$ pair

The initial partition is divided to form the final one, as shown in the figure below:

![Diagram of partitions](image)

**Figure 5**: Partition $P_0$ (left) and $P$ (right). We call this $P_0$ corner patch.

Then, we define the spaces:

$$Q_{1,P} := \left\{ v \in H^1_0(\Omega)^2 : v \circ F_K \in Q_1(K)^2 \quad \forall K \in P \right\},$$

and

$$M_P := \left\{ q \in L^2_0(\Omega) : q \circ F_K \in P_0(K) \quad \forall K \in P \right\}.$$
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![Partition Diagram]

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and

$$M_P := \{ q \in L_0^2(\Omega) : q \circ F_K \in P_0(K) \quad \forall K \in P \}.$$
A direct consequence of the first part:

**Lemma**

*For the subspace* $G \subset \mathcal{M}_{P_0} \subset \mathcal{M}_P$, *defined by*

$$G := \{ q \in \mathcal{M}_{P_0} : [q]_{\gamma_c} = 0 \text{ for } \gamma_c \in \mathcal{E}_c \},$$

*there exists a constant* $\beta_G$ *independent of aspect ratios such that*

$$\sup_{\nu \in Q_{1,P}} \frac{\langle \nabla \cdot \nu, q \rangle_{\Omega}}{|\nu|_{1,\Omega}} \geq \beta_G \| q \|_{0,\Omega} \quad \text{for all } q \in G.$$
The \( \mathbb{Q}_1 \times \mathbb{P}_0 \) pair

The method: Find \((u_\mathcal{P}, p_\mathcal{P}) \in \mathbb{Q}_1, \mathcal{P} \times \mathcal{M}_\mathcal{P}\) such that

\[
B_s((u_\mathcal{P}, p_\mathcal{P}), (v, q)) = (f, v)_\Omega \quad \text{for all } (v, q) \in \mathbb{Q}_1, \mathcal{P} \times \mathcal{M}_\mathcal{P}.
\]

Here,

\[
B_s((u, p), (v, q)) = B((u, p), (v, q)) - \frac{1}{4} \tilde{S}(p; q),
\]

the stabilisation terms are

\[
\tilde{S}(p, q) := \sum_{M \in \mathcal{P}_0} S_M(p, q) + \sum_{\gamma_c \in \mathcal{E}_c} S_{\gamma_c}(p, q),
\]

and

\[
S_M(p, q) := \sum_{e \in \mathcal{E}_M} \frac{|K|}{|e|} \int_{[p][q]} \quad \text{and} \quad S_{\gamma_c}(p, q) := \sum_{e \subset \gamma_c} \frac{\min\{|K|, |K'|\}}{|e|} \int_{[p][q]}.
\]

Remark: Without the terms \( S_{\gamma_c} \) the method has been proposed by L& S.
The $Q_1 \times P_0$ pair

The method: Find $(u_P, p_P) \in Q_{1,P} \times M_P$ such that

$$B_s((u_P, p_P), (v, q)) = (f, v)_\Omega$$

for all $(v, q) \in Q_{1,P} \times M_P$.

Here,

$$B_s((u, p), (v, q)) = B((u, p), (v, q)) - \frac{1}{4} \tilde{S}(p; q),$$

the stabilisation terms are

$$\tilde{S}(p, q) := \sum_{M \in P_0} S_M(p, q) + \sum_{\gamma_c \in E_c} S_{\gamma_c}(p, q),$$

and

$$S_M(p, q) := \sum_{e \in E_M} \frac{|K|}{|e|} \int_e [p][q] \quad \text{and} \quad S_{\gamma_c}(p, q) := \sum_{e \subset \gamma_c} \frac{\min\{|K|, |K'|\}}{|e|} \int_e [p][q].$$

Remark: Without the terms $S_{\gamma_c}$ the method has been proposed by L& S.
The $Q_1 \times P_0$ pair

**Theorem**

The stabilising terms $S_M$ and $S_{\gamma_c}$ control all the unstable modes. Then, there exists a constant $\mu_s > 0$ independent of the aspect ratio $\varrho$, such that

$$\sup_{(v, q) \in Q_1, P \times M_P} \frac{B_s((u, r)(v, q))}{\| (v, q) \|} \geq \mu_s \| (w, r) \| \quad \text{for all } (w, r) \in Q_1, P \times M_P.$$

Moreover, if $p \in H^1(\Omega)$, then there exists a positive constant $C$ such that

$$\left( \inf_{(v, P, q) \in Q_1, P \times M_P} \| (u - v, p - q) \| + \sum_{K \in P} h_{K,x} \| \partial_x p \|_{0,K} + h_{K,y} \| \partial_y p \|_{0,K} \right) \leq (1 + C\mu_s^{-1})$$

where $h_{K,x}$ and $h_{K,y}$ are the diameters of $K \in P$ in the $x$- and $y$-directions, respectively.
The $Q_1 \times P_0$ pair

Figure 6: Stability constants $\mu_s$, and the LS method for a T-mesh.
The Oseen equation

Find a pair \((u, p)\) such that \(u = 0\) on \(\partial \Omega\), and

\[-\nu \Delta u + b \cdot \nabla u + \sigma u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \Omega,
\]

where \(\sigma, \nu > 0\) and \(\nabla \cdot b = 0\) in \(\Omega\).

Remark: We use the same finite element spaces as before. So, the stabilisation mechanisms for the pressure are identical.
The Oseen equation

The stabilised finite element method: Find \((u_\mathcal{P}, p_\mathcal{P}) \in Q_{1,\mathcal{P}} \times M_\mathcal{P}\) such that:

\[
B((u_\mathcal{P}, p_\mathcal{P}), (v, q)) + s_v(u_\mathcal{P}, v) - \alpha_p s_p(p, q) = (f, v)_\Omega \quad \forall (v, q) \in Q_{1,\mathcal{P}} \times M_\mathcal{P},
\]

where
The Oseen equation

The stabilised finite element method: Find \((\mathbf{u}_P, p_P) \in Q_1, P \times M_P\) such that:

\[
B((\mathbf{u}_P, p_P), (\mathbf{v}, q)) + s_v(\mathbf{u}_P, \mathbf{v}) - \alpha_p s_p(p, q) = (f, \mathbf{v})_\Omega \quad \forall (\mathbf{v}, q) \in Q_1, P \times M_P,
\]

where

\[
B((\mathbf{u}, p), (\mathbf{v}, q)) := \nu(\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega + (\mathbf{b} \cdot \nabla \mathbf{u}, \mathbf{v})_\Omega + \sigma(\mathbf{u}, \mathbf{v})_\Omega - (p, \nabla \cdot \mathbf{v})_\Omega - (q, \nabla \cdot \mathbf{u})_\Omega,
\]

\[=: a(\mathbf{u}, \mathbf{v})\]
The stabilised finite element method: Find \((u_P, p_P) \in Q_1, P \times M_P\) such that:

\[
B((u_P, p_P), (v, q)) + s_v(u_P, v) - \alpha_p s_p(p, q) = (f, v)_\Omega \quad \forall (v, q) \in Q_1, P \times M_P,
\]

where

\[
s_p(p, q) = \text{exactly as before},
\]
The Oseen equation

The stabilised finite element method: Find \((u_\mathcal{P}, p_\mathcal{P}) \in Q_{1,\mathcal{P}} \times M_\mathcal{P}\) such that:

\[
B((u_\mathcal{P}, p_\mathcal{P}), (v, q)) + s_v(u_\mathcal{P}, v) - \alpha_p s_p(p, q) = (f, v)_\Omega \quad \forall (v, q) \in Q_{1,\mathcal{P}} \times M_\mathcal{P},
\]

where

\(s_v(u, v)\) is symmetric and needs to satisfy: Let \(\|v\|_s := s_v(v, v)\). We assume:

\[
\begin{align*}
& s_v(w, v) \leq c_s \|w\|_s |v|_{1, \Omega}, \\
& s_v(v, v) \geq 0, \\
& \sum_{K \in \mathcal{P}} \gamma_K \|\kappa_K (\nabla \cdot v)\|_{0,K}^2 \leq s_v(v, v),
\end{align*}
\]

for all \(v, w \in H_0^1(\Omega)^2\).
The Oseen equation

The stabilised finite element method: Find \((u_\mathcal{P}, p_\mathcal{P}) \in Q_{1,\mathcal{P}} \times M_\mathcal{P}\) such that:

\[
B((u_\mathcal{P}, p_\mathcal{P}), (v, q)) + s_v(u_\mathcal{P}, v) - \alpha p s_p(p, q) = (f, v)_\Omega \quad \forall (v, q) \in Q_{1,\mathcal{P}} \times M_\mathcal{P},
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where

\(s_v(u, v)\) is symmetric and needs to satisfy: Let \(\|v\|_s := s_v(v, v)\). We assume:

\[
\begin{align*}
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    s_v(v, v) &\geq 0, \\
    \sum_{K \in \mathcal{P}} \gamma_K \|\kappa_K (\nabla \cdot v)\|_{0,K}^2 &\leq s_v(v, v),
\end{align*}
\]

for all \(v, w \in H_0^1(\Omega)^2\). Using these conditions, we take \(\alpha_p \geq \alpha := (c_a^2 + c_s^2)^{-1}\).
**Lemma**

Let $s_v$ satisfy the previous assumptions. Let us define the mesh-dependent norm

$$\| (v, q) \| := \| v \|_{a+s}^2 + \alpha \| q \|_{0, \Omega}^2 + s_p (q, q).$$

Then, there exist $\mu_s > 0$, independent of the aspect ratio of the mesh, and of $\nu$, such that:

$$\sup_{(v, q) \in Q_{1, \mathcal{P}} \times \mathcal{M}_\mathcal{P}} \frac{B_s ((w, r), (v, q))}{\| (v, q) \|} \geq \mu_s \| (w, r) \| \quad \text{for all } (w, r) \in Q_{1, \mathcal{P}} \times \mathcal{M}_\mathcal{P},$$

where $\mu_s = \beta_G^2 / [2(1 + \beta_G)(17 + 16\beta_G)]$

Moreover, error estimates in the triple norm, with constants independent of $\nu$, can be proved.
The Oseen equation

Lemma

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Moreover, error estimates in the triple norm, with constants independent of $\nu$, can be proved.
Some numerics for Oseen: some concrete choices for $s_v$

Defining $\kappa_D := \text{id} - \Pi_0^D$, we can define the following stabilising terms:

\[
\begin{align*}
    s_1^1(u, v) &:= \sum_{M \in \mathcal{P}_0} \gamma_M (\kappa_M(\nabla \cdot u), \kappa_M(\nabla \cdot v))_M \\
    &\quad + \sum_{K \in \mathcal{P}} (\kappa_K(b_K \cdot \nabla u), \kappa_K(b_K \cdot \nabla v))_K , \\
    s_2^2(u, v) &:= \sum_{M \in \mathcal{P}_0} \delta_x (\kappa_M(\partial_x u), \partial_x v)_M + \delta_y (\kappa_M(\partial_y u), \partial_y v)_M \quad \text{(LPS)} ,
\end{align*}
\]

where
Defining $\kappa_D := id - \Pi_0^D$, we can define the following stabilising terms:

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$$+ \sum_{K \in \mathcal{P}} (\kappa_K (b_K \cdot \nabla u), \kappa_K (b_K \cdot \nabla v))_K,$$

$$s_v^2(u, v) := \sum_{M \in \mathcal{P}_0} \delta_x (\kappa_M (\partial_x u), \partial_x v)_M + \delta_y (\kappa_M (\partial_y u), \partial_y v)_M \quad \text{(LPS)},$$

where

$$\delta_{K,x} := \nu^{-1} \|b\|_{\infty,K}^2 h_{K,x}^2 \min\{1, \text{Pe}_{\min,K}^{-1}\}$$

$$\delta_{K,y} := \nu^{-1} \|b\|_{\infty,K}^2 h_{K,y}^2 \min\{1, \text{Pe}_{\min,K}^{-1}\}$$

$$\text{Pe}_{\min,K} := \nu^{-1} \min\{h_{K,x}, h_{K,y}\} \|b\|_{\infty,K}$$
Some numerics for Oseen: some concrete choices for $s_v$

Defining $\kappa_D := id - \Pi_D^0$, we can define the following stabilising terms:

$$s^1_v(u, v) := \sum_{M \in \mathcal{P}_0} \gamma_M (\kappa_M(\nabla \cdot u), \kappa_M(\nabla \cdot v))_M$$

$$+ \sum_{K \in \mathcal{P}} (\kappa_K(b_K \cdot \nabla u), \kappa_K(b_K \cdot \nabla v))_K,$$

$$s^2_v(u, v) := \sum_{M \in \mathcal{P}_0} \delta_x (\kappa_M(\partial_x u), \partial_x v)_M + \delta_y (\kappa_M(\partial_y u), \partial_y v)_M \quad \text{(LPS)},$$

where

$$\gamma_M := \max\{1, Pe_{\mathcal{P}_0}^{\min}\} \quad \text{where} \quad Pe_{\mathcal{P}_0}^{\min} := \min_{M \in \mathcal{P}_0} \frac{\|b\|_{\infty, M}}{\nu} \min\{h_x, M, h_y, M\},$$

or

$$\gamma_M := 1 + ind(M) Pe_M^{\min} \quad \text{where} \quad ind(M) := 1 - \frac{\rho_M|M|}{\max_{M \in \mathcal{P}_0} |M|}.$$
Some numerics for Oseen: some concrete choices for $s_\nu$

The mesh:

![A Shishkin mesh with parameter $\lambda = \min\{\frac{1}{2}, 2\nu \ln N\}$ ($\nu = 1/32$), with $N = 8$ intervals.](image)

**Figure 7**: A Shishkin mesh with parameter $\lambda = \min\{\frac{1}{2}, 2\nu \ln N\}$ ($\nu = 1/32$), with $N = 8$ intervals.
Some numerics for Oseen: some concrete choices for $s_v$

The solution:

Figure 8: Nodal interpolation of $u_1$ (left) and $L^2$ projection of $p$ (right) for $\nu = 10^{-6}$. 
Some error results $(\nu = 10^{-6})$: We define, for a given partition $\mathcal{P}$, the relative errors

$$ E_{p}^{\text{rel}} := \frac{\| p - p_{\mathcal{P}} \|_{0,\Omega}}{\| p - \Pi p \|_{0,\Omega}} \quad \text{and} \quad E_{u}^{\text{rel}} := \frac{\| u - u_{\mathcal{P}} \|_{1,\Omega}}{\| u - I_{\mathcal{P}} u \|_{1,\Omega}}. $$

Table 1: Here, $N = 8$, $\lambda = 0.01$, and $Q_{1,\mathcal{P}} \times G$.

<table>
<thead>
<tr>
<th>$s_{v}$</th>
<th>$\gamma_M$</th>
<th>$E_{p}^{\text{rel}}$</th>
<th>$E_{u}^{\text{rel}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{v}^{1}$</td>
<td>First</td>
<td>25.02</td>
<td>1.0</td>
</tr>
<tr>
<td>$s_{v}^{1}$</td>
<td>Second</td>
<td>25.31</td>
<td>1.0</td>
</tr>
<tr>
<td>LPS</td>
<td>-</td>
<td>1.40</td>
<td>1.0002</td>
</tr>
</tbody>
</table>
Some numerics for Oseen: some concrete choices for $s_v$

Some error results ($\nu = 10^{-6}$): We define, for a given partition $\mathcal{P}$, the relative errors

$$E_{p}^{\text{rel}} := \frac{\|p - p_{\mathcal{P}}\|_{0, \Omega}}{\|p - \Pi p\|_{0, \Omega}}, \quad \text{and} \quad E_{u}^{\text{rel}} := \frac{|u - u_{\mathcal{P}}|_{1, \Omega}}{|u - I_{\mathcal{P}} u|_{1, \Omega}}.$$

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<th>$E_{u}^{\text{rel}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_v^1$</td>
<td>First</td>
<td>1.06</td>
<td>1.0</td>
</tr>
<tr>
<td>$s_v$</td>
<td>Second</td>
<td>1.06</td>
<td>1.0</td>
</tr>
<tr>
<td>LPS</td>
<td>-</td>
<td>1.22</td>
<td>1.0219</td>
</tr>
</tbody>
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Some numerics for Oseen: some concrete choices for $s_v$

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Table 1: Here, $N = 8$, $\lambda = 0.01$, and $Q_{1,\mathcal{P}} \times M_{\mathcal{P}}$, and $\alpha_{p} = 1$. 

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<tr>
<th>$s_v$</th>
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<th>$E_{p}^{\text{rel}}$</th>
<th>$E_{u}^{\text{rel}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{v}^1$</td>
<td>First</td>
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<tr>
<td>$s_{v}^1$</td>
<td>Second</td>
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<td>1.0</td>
</tr>
<tr>
<td>LPS</td>
<td>-</td>
<td>7.47</td>
<td>1.0002</td>
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<tbody>
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<td>1.0</td>
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<tr>
<td>$s_{v}^{1}$</td>
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<td>1.0</td>
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<td>-</td>
<td>6.73</td>
<td>1.0152</td>
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Some numerics for Oseen: some concrete choices for $s_v$

Sharpeness of the layers with varying $\lambda$.

<table>
<thead>
<tr>
<th>$s_v, \gamma_M$</th>
<th>$\lambda = 0.5000$</th>
<th>$\lambda = 1.0000 \cdot 10^{-2}$</th>
<th>$\lambda = 1.0000 \cdot 10^{-4}$</th>
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<tbody>
<tr>
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**Figure 9:** Meshes: $N = 8$, $\lambda$ (left to right). Using $Q_{1,p} \times G$. 
Some numerics for Oseen: some concrete choices for $s_v$

Sharpness of the layers with varying $\lambda$.

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$s^1_v + \text{First}$

Figure 9:  Meshes: $N = 8$, $\lambda$ (left to right). Using $Q_1,p \times G$. 
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</tr>
</thead>
<tbody>
<tr>
<td>$s_v^1 + \text{Second}$</td>
<td><img src="image1" alt="Graph1" /></td>
<td><img src="image2" alt="Graph2" /></td>
<td><img src="image3" alt="Graph3" /></td>
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**Figure 10**: Meshes: $N = 8$, $\lambda$ (left to right). Using $Q_{1,p} \times \mathcal{M}_p$, $\alpha_p = 1$. 
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<tr>
<td>$s_v^1 + \text{First}$</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
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Figure 10: Meshes: $N = 8, \lambda$ (left to right). Using $Q_{1,p} \times M_P$, $\alpha_p = 1$. 
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Sharpness of the layers with varying $\lambda$.

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Figure 10: Meshes: $N = 8, \lambda$ (left to right). Using $Q_{1,p} \times M_P, \alpha_p = 1$. 
Conclusions and perspectives

1. Identification of the minimal number of spurious pressure modes on anisotropic meshes.
2. A new family of inf-sup stable finite element spaces. These enjoy the same approximation properties of the original one.
3. A stabilised variant penalising these modes in the formulation: stability and optimal convergence.
4. Extension to the (optimal) $Q_1^2 \times P_0$ pair, and Oseen.

Perspectives and open questions:
- Adaptivity?
- Triangles?
- Continuous pressures?
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