Stabilised finite element methods in anisotropic quadrilateral meshes

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Outline

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3. The $\mathbb{Q}_{k+1}^2 \times \mathbb{P}_{k-1}$ pair.
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    - The new mixed method.
    - The stabilised variant.
4. The $\mathbb{Q}_1^2 \times \mathbb{P}_0$ pair.
5. The Oseen equation.
6. Conclusions, open questions.

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Motivation: A saddle point problem

The setting : Find $(u, p) \in \boldsymbol{V} \times Q$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= F(v) \qquad \forall\, v \in \boldsymbol{V}\,, \\ b(u,q) &= 0 \qquad \forall\, q \in Q\,. \end{aligned}$$

The Galerkin scheme : Given $\boldsymbol{V}_h \subset \boldsymbol{V}$ and $Q_h \subset Q$, finite-dimensional spaces: Find $(u_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

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Remark: The stability and error estimates constants vary as β^{-2}, where β is the discrete inf-sup constant.

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Problem: In many interesting cases, β degenerates with some important quantity (e.g., the aspect ratio). Fortunately, in some cases, the following decomposition can be proved:

$$Q_h = Q_h^G \oplus Q_h^B \,,$$

where $\boldsymbol{V}_h \times Q_h^G$ satisfies:

$$\sup_{v_h \in \boldsymbol{V}_h \setminus \{0\}} \frac{b(v_h, q_h)}{\|v_h\|_{\boldsymbol{V}}} \geq \beta_G \, \|q_h\|_Q \qquad \forall \, q_h \in Q_h^G \,,$$

where $\beta_G > 0$ does not depend on any bad parameter.
Then, the following weak inf-sup condition can be proved:

$$\sup_{v_h \in \boldsymbol{V}_h \setminus \{0\}} \frac{b(v_h, q_h)}{\|v_h\|_{\boldsymbol{V}}} \geq \beta_G \, \|q_h\|_Q - C\|q_h - \Pi q_h\|_Q \qquad \forall \, q_h \in Q_h \,,$$

where $\Pi : Q \to Q_h^G$ is any continuous linear projection onto the good space Q_h^G, and $C > 0$ is an $O(1)$ constant.

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Two possible solutions:

- If Q_h^G is essentially equal to Q_h, just use the pair $\boldsymbol{V}_h \times Q_h^G$ as a mixed method; else
- A stabilised variant, in the vein of the *minimal stabilisation* (cf. Brezzi & Fortin): Find $(u_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

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For both variants, stability and convergence, with constants depending only on β_G, can be proved.

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The Stokes problem

The Stokes problem : Find a pair $(\boldsymbol{u}, p)$ such that $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$, and

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad , \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega\,.$$

The finite element spaces : For a given partition $\mathcal{P}$ and an integer $k \geq 1$ we define:

$$\boldsymbol{V}_{\mathcal{P}} = \{\boldsymbol{v} \in H_0^1(\Omega)^2 : \boldsymbol{v} \circ F_K \in \mathbb{Q}_{k+1}^2 \quad \forall K \in \mathcal{P}\}$$
$$\mathcal{M}_{\mathcal{P}} = \{q \in L_0^2(\Omega) : q \circ F_K \in \mathbb{P}_{k-1} \quad \forall K \in \mathcal{P}\}$$

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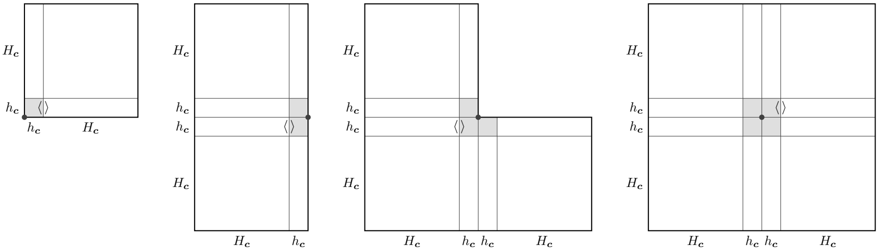
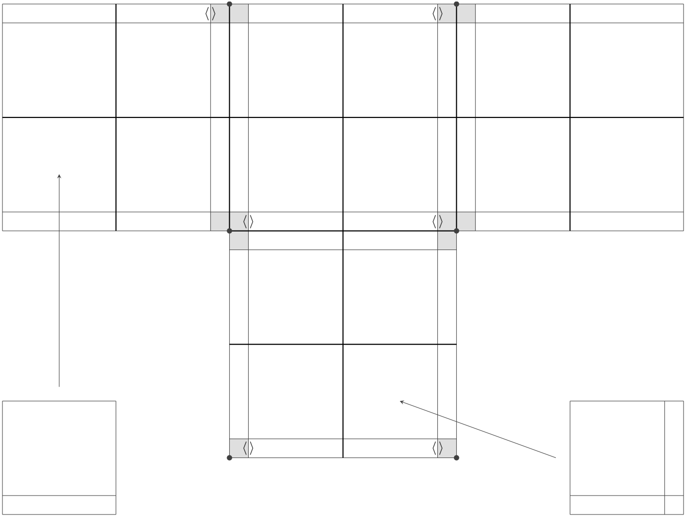


Figure 1 : Typical anisotropically refined corner patches Ω_c with the corresponding subsets ω_c shown shaded. On the left: a single corner patch.

The Stokes problem



(a) Edge patch

(b) Corner patch

Figure 2 : A typical example of an anisotropically refined mesh.

The Stokes problem

Lemma (A& C 2000)

There exists $C > 0$ *such that*

$$\inf_{q\in\mathcal{M}_{\mathcal{P}}} \sup_{\boldsymbol{v}\in\boldsymbol{V}_{\mathcal{P}}} \frac{(\nabla\cdot\boldsymbol{v},q)_\Omega}{|\boldsymbol{v}|_{1,\Omega}\|q\|_{0,\Omega}} = \beta_{\mathcal{P}} = Ck^{-1/2}\min\{1,k\sqrt{\varrho}\}\,,$$

where $\varrho = h_c/H_c$ *is the mesh aspect ratio.*

The Stokes problem

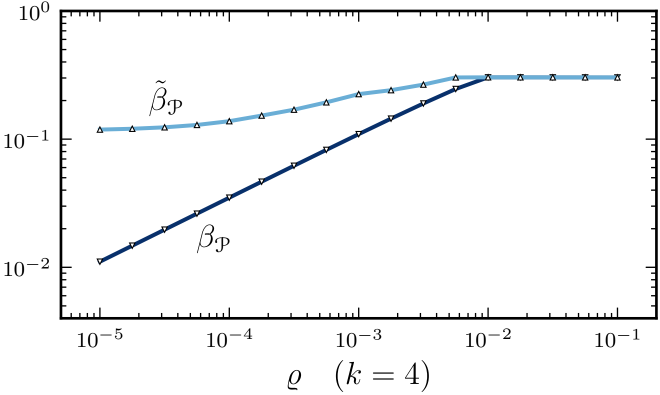
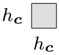
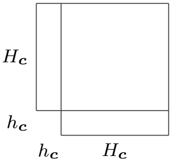
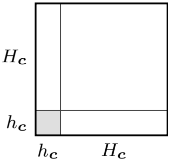


Figure 3 : Behaviour of the inf-sup constants $\beta_{\mathcal{P}}$ and $\tilde{\beta}_{\mathcal{P}}$ with respect to the aspect ratio and polynomial degree $k = 4$ on the T-mesh.



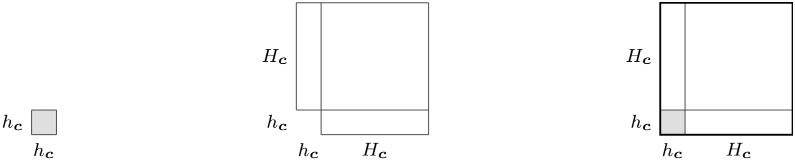




$\beta_{\mathcal{P}_a} = C$ $\qquad$ $\beta_{\mathcal{P}_b} = Ck^{-1/2}$ $\qquad$ $\beta_{\mathcal{P}} \sim \sqrt{\varrho}$

Together, ...

Whose fault is that?



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Together, ... $\mathcal{M}_{\mathcal{P}} = \mathcal{M}_{\mathcal{P}_a} \oplus \mathcal{M}_{\mathcal{P}_b} \oplus \mathrm{Span}\{q_B^c\}$ where

$$q_B^c = \begin{cases} 1 & \text{in } \omega_c, \\ -\frac{|\omega_c|}{|\Omega_c \setminus \omega_c|} & \text{in } \Omega_c \setminus \omega_c, \end{cases}$$

and ω_c is the shaded, extremely small subdomain.

Whose fault is that?

Theorem (Corollary of A& C 2000)

Let $\mathcal{M}_{\mathcal{P}}^* = \{q \in \mathcal{M}_{\mathcal{P}} : \int_{\omega_c} q = 0, \quad$ *for all corner patches*$\}$. *Then, there is a positive constant* C, *independent of any aspect ratio such that :*

$$\inf_{q \in \mathcal{M}_{\mathcal{P}}^*} \sup_{\boldsymbol{v} \in \boldsymbol{V}_{\mathcal{P}}} \frac{(\nabla \cdot \boldsymbol{v}, q)_\Omega}{|\boldsymbol{v}|_{1,\Omega} \|q\|_{0,\Omega}} \geq C k^{-1/2} .$$

Consequence 1 : The pair $\boldsymbol{V}_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}^*$ is a uniformly inf-sup stable pair.

Consequence 2 : Confirmation that the culprit of the inf-sup defficiency is only one pressure mode per corner patch. Namely, the function q_B^c defined previously. Then, it is very easy to propose a stabilised finite element method using a minimal stabilisation idea.

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First Solution

The first stabilised method : Find $(\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}) \in \boldsymbol{V}_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$ such that:

$$B((\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}), (\boldsymbol{v}, q)) - \tilde{s}(p_{\mathcal{P}}, q) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}\,,$$

where

$$\begin{aligned} B((\boldsymbol{u}, p), (\boldsymbol{v}, q)) :=& (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega - (p, \nabla \cdot \boldsymbol{v})_\Omega - (q, \nabla \cdot \boldsymbol{u})_\Omega\,, \\ \tilde{s}(p, q) :=& ((I - \Pi)p, (I - \Pi)q)_\Omega\,. \end{aligned}$$

- Pros: Stability can be proved quite easily, and numerics follow.
- Cons: The consistency error can not be bounded optimally.

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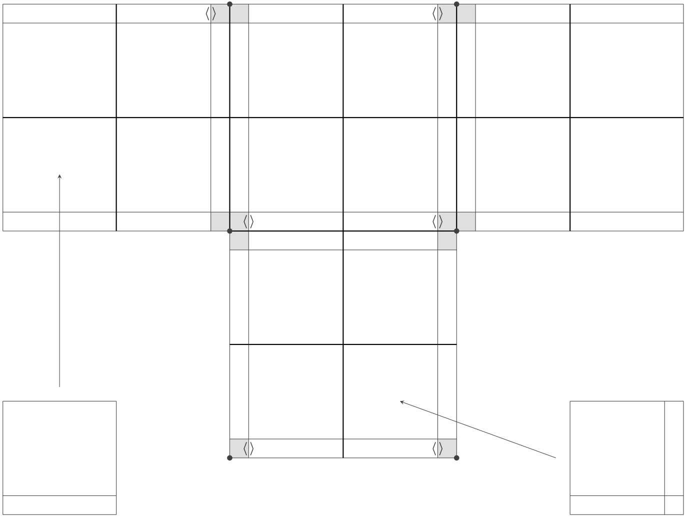
The new mixed method

Important Remark : There exists a projection operator $\tilde{\Pi}$ such that

$$\tilde{s}(p,q) = ((I-\tilde{\Pi})p,(I-\tilde{\Pi})q)_\Omega = \frac{\tau_{\gamma_{\boldsymbol{c}}}}{k^2}\int_{\gamma_{\boldsymbol{c}}}[\![p]\!]\int_{\gamma_{\boldsymbol{c}}}[\![q]\!]\,,$$

where $\tau_{\gamma_{\boldsymbol{c}}} > 0$ is an appropriate constant, and $\gamma_{\boldsymbol{c}}$ is a single, arbitrary, edge connecting a small square element $\kappa_{\boldsymbol{c}}$ in $\Omega_{\boldsymbol{c}}$ with a stretched element $K_{\boldsymbol{c}}$.

Reminder of the T-mesh



(a) Edge patch

(b) Corner patch

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The useful consequence :

Theorem

Let $\widetilde{\mathcal{M}}_{\mathcal{P}} \subset \mathcal{M}_{\mathcal{P}}$ *denote the subspace defined by*

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Then, the following inf-sup stability holds

$$\inf_{q \in \tilde{\mathcal{M}}_{\mathcal{P}}} \sup_{\boldsymbol{v} \in \boldsymbol{V}_{\mathcal{P}}} \frac{(\nabla \cdot \boldsymbol{v}, q)_{\Omega}}{|\boldsymbol{v}|_{1,\Omega} \|q\|_{0,\Omega}} \geq \tilde{\beta}_{\mathcal{P}} > 0,$$

where

$$\tilde{\beta}_{\mathcal{P}} = \max\{\beta_{\mathcal{P}}, C\, k^{-3/2}\} .$$

Moreover, if $p \in H^1(\Omega)$*, then there exists a positive constant* C *such that*

$$\inf_{\tilde{q}_{\mathcal{P}} \in \tilde{\mathcal{M}}_{\mathcal{P}}} \|p - \tilde{q}_{\mathcal{P}}\|_{0,\Omega}^2 \leq C \inf_{q_{\mathcal{P}} \in \mathcal{M}_{\mathcal{P}}} \left(\|p - q_{\mathcal{P}}\|_{0,\Omega}^2 + \sum_{\boldsymbol{c}} \frac{|\gamma_{\boldsymbol{c}}|^2}{k^2} \|\partial_{n_{\boldsymbol{c}}}(p - q_{\mathcal{P}})\|_{0,\kappa_{\boldsymbol{c}} \cup K_{\boldsymbol{c}}}^2\right) .$$

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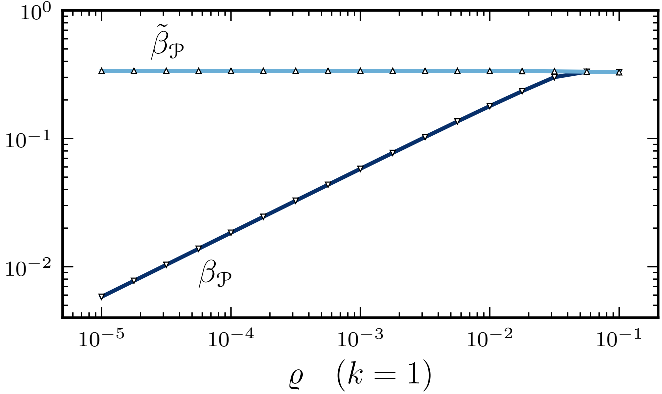
The new mixed method: Some numerics



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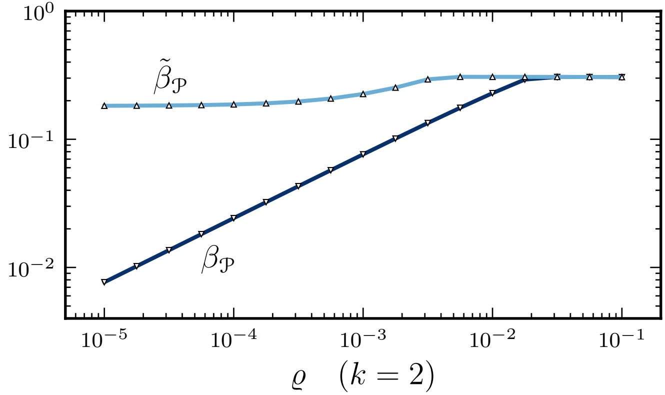

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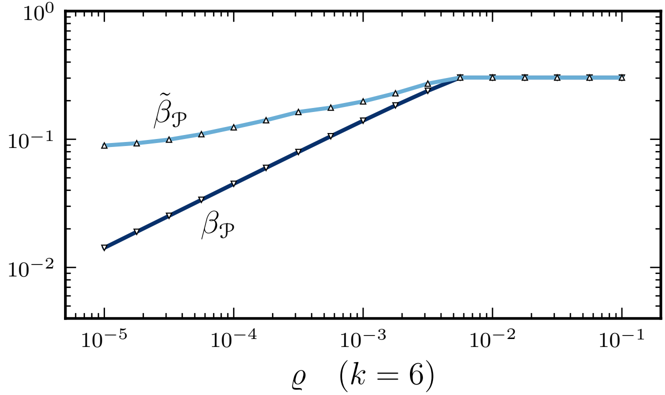

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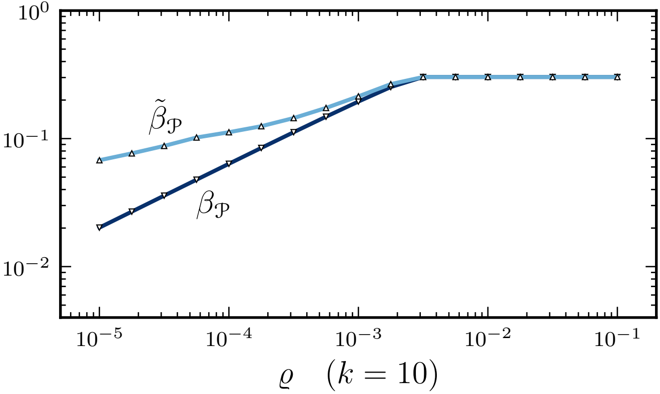

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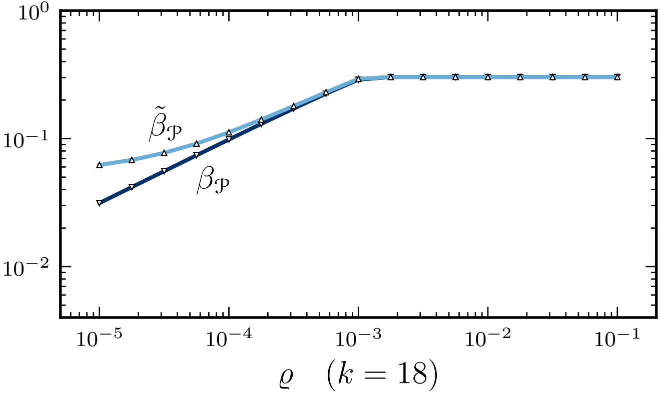

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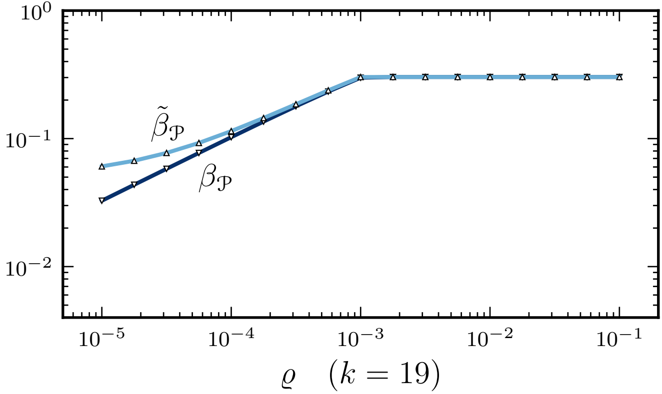

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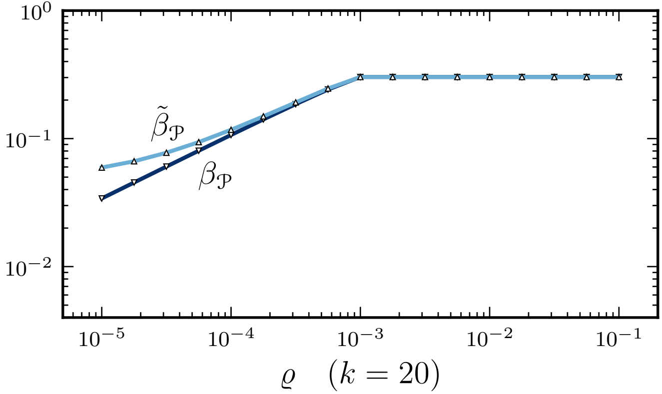

The stabilised alternative

The stabilised method : Find $(\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}) \in \boldsymbol{V}_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$ such that:

$$B((\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}), (\boldsymbol{v}, q)) - S(p_{\mathcal{P}}, q) = (\boldsymbol{f}, \boldsymbol{v})_{\Omega} \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}},$$

where

$$S(p, q) := \frac{1}{k^2} \sum_{\boldsymbol{c}} \int_{\gamma_{\boldsymbol{c}}} [\![p]\!] \cdot \int_{\gamma_{\boldsymbol{c}}} [\![q]\!] .$$

Lemma

There exist positive constants C_1, C_2 such that for all $q_{\mathcal{P}} \in \mathcal{M}_{\mathcal{P}}$,

$$C_1 \|q_{\mathcal{P}} - \tilde{\Pi}_{\mathcal{P}} q_{\mathcal{P}}\|_{\Omega}^2 \leq S(q_{\mathcal{P}}, q_{\mathcal{P}}) \leq C_2 \|q_{\mathcal{P}} - \tilde{\Pi}_{\mathcal{P}} q_{\mathcal{P}}\|_{\Omega}^2 .$$

Furthermore,

$$S(q_{\mathcal{P}}, q_{\mathcal{P}}) \leq C \begin{cases} \|q_{\mathcal{P}}\|_{0,\Omega}^2 \\ k^{-2} \sum_{\boldsymbol{c}} \left(\|p - q_{\mathcal{P}}\|_{0,\kappa_{\boldsymbol{c}} \cup K_{\boldsymbol{c}}}^2 + |\gamma_{\boldsymbol{c}}|^2 \|\partial_{n_{\boldsymbol{c}}} (p - q_{\mathcal{P}})\|_{\kappa_{\boldsymbol{c}} \cup K_{\boldsymbol{c}}}^2 \right) \end{cases}$$

for all $p \in H^1(\Omega)$.

The stabilised alternative

The stabilised method : Find $(\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}) \in \boldsymbol{V}_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$ such that:

$$B((\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}), (\boldsymbol{v}, q)) - S(p_{\mathcal{P}}, q) = (\boldsymbol{f}, \boldsymbol{v})_{\Omega} \qquad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}\,,$$

where

$$S(p,q) := \frac{1}{k^2} \sum_{\boldsymbol{c}} \int_{\gamma_{\boldsymbol{c}}} [\![p]\!] \cdot \int_{\gamma_{\boldsymbol{c}}} [\![q]\!]\,.$$

Lemma

There exist positive constants C_1, C_2 such that for all $q_{\mathcal{P}} \in \mathcal{M}_{\mathcal{P}}$,

$$C_1 \|q_{\mathcal{P}} - \tilde{\Pi}_{\mathcal{P}} q_{\mathcal{P}}\|_{\Omega}^2 \leq S(q_{\mathcal{P}}, q_{\mathcal{P}}) \leq C_2 \|q_{\mathcal{P}} - \tilde{\Pi}_{\mathcal{P}} q_{\mathcal{P}}\|_{\Omega}^2.$$

Furthermore,

$$S(q_{\mathcal{P}}, q_{\mathcal{P}}) \leq C \begin{cases} \|q_{\mathcal{P}}\|_{0,\Omega}^2 \\ k^{-2} \sum_{\boldsymbol{c}} \left(\|p - q_{\mathcal{P}}\|_{0,\kappa_{\boldsymbol{c}} \cup K_{\boldsymbol{c}}}^2 + |\gamma_{\boldsymbol{c}}|^2 \|\partial_{n_{\boldsymbol{c}}}(p - q_{\mathcal{P}})\|_{\kappa_{\boldsymbol{c}} \cup K_{\boldsymbol{c}}}^2 \right) \end{cases}$$

for all $p \in H^1(\Omega)$.

The stabilised alternative

Theorem

For all $(\boldsymbol{w}, r) \in \boldsymbol{V}_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$*, there holds*

$$\sup_{(\boldsymbol{v},q)\in \boldsymbol{V}_{\mathcal{P}}\times\mathcal{M}_{\mathcal{P}}} \frac{B_s((\boldsymbol{w},r),(\boldsymbol{v},q))}{|||(\boldsymbol{v},q)|||} \geq C\tilde{\beta}_{\mathcal{P}}^2 \, |||(\boldsymbol{w},r)||| \, .$$

Moreover, if $p \in H^1(\Omega)$*, then there exists a positive constant* C *such that*

$$|||(\boldsymbol{u}-\boldsymbol{u}_{\mathcal{P}}^s, p-p_{\mathcal{P}}^s)||| \leq (1 + C\tilde{\beta}_{\mathcal{P}}^{-2})$$
$$\inf_{(\boldsymbol{v}_{\mathcal{P}},p_{\mathcal{P}})\in \boldsymbol{V}_{\mathcal{P}}\times\mathcal{M}_{\mathcal{P}}} \left\{ |||(\boldsymbol{u}-\boldsymbol{v}_{\mathcal{P}}, p-q_{\mathcal{P}})||| + k^{-1}\Big(\sum_{\boldsymbol{c}} |\gamma_{\boldsymbol{c}}|^2 \|\partial_{n_{\boldsymbol{c}}}(p-q_{\mathcal{P}})\|_{0,\kappa_{\boldsymbol{c}}\cup K_{\boldsymbol{c}}}^2\Big)^{\frac{1}{2}} \right\} .$$

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The stabilised alternative

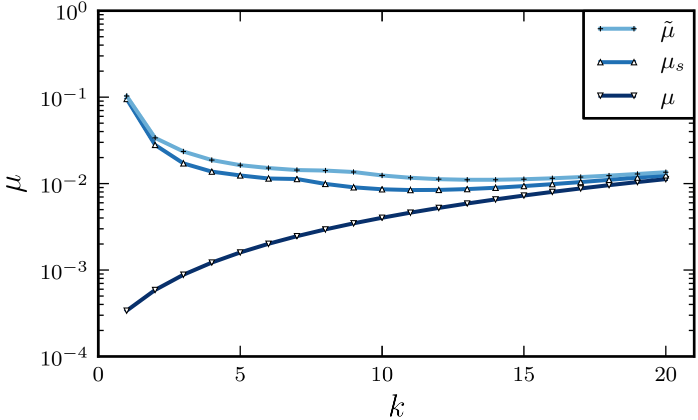

Figure 4 : Behaviour of inf-sup constants of different methods on the T-mesh shown in Figure 1 for fixed aspect ratio $\varrho = 10^{-4}$ and increasing polynomial degree k.

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The $\mathbb{Q}_1 \times \mathbb{P}_0$ pair

The initial partition is divided to form the final one, as shown in the figure below:

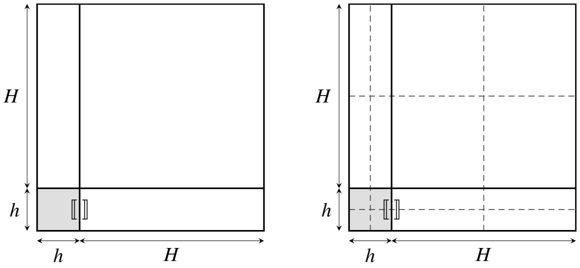

Figure 5 : Partition $\mathcal{P}_0$ (left) and $\mathcal{P}$ (right). We call this $\mathcal{P}_0$ corner patch.

Then, we define the spaces:

$$\boldsymbol{Q}_{1,\mathcal{P}} := \{\boldsymbol{v} \in H_0^1(\Omega)^2 : \boldsymbol{v} \circ F_K \in \mathbb{Q}_1(K)^2 \quad \forall K \in \mathcal{P}\},$$

and

$$\mathcal{M}_{\mathcal{P}} := \{q \in L_0^2(\Omega) : q \circ F_K \in \mathbb{P}_0(K) \quad \forall K \in \mathcal{P}\}.$$

The $\mathbb{Q}_1 \times \mathbb{P}_0$ pair

The initial partition is divided to form the final one, as shown in the figure below:

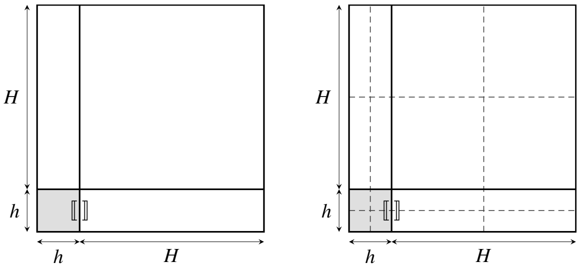

Figure 5 : Partition $\mathcal{P}_0$ (left) and $\mathcal{P}$ (right). We call this $\mathcal{P}_0$ corner patch.

Then, we define the spaces:

$$\boldsymbol{Q}_{1,\mathcal{P}} := \{\boldsymbol{v} \in H_0^1(\Omega)^2 : \boldsymbol{v} \circ F_K \in \mathbb{Q}_1(K)^2 \quad \forall K \in \mathcal{P}\},$$

and

$$\mathcal{M}_{\mathcal{P}} := \{q \in L_0^2(\Omega) \colon q \circ F_K \in \mathbb{P}_0(K) \quad \forall K \in \mathcal{P}\}.$$

The $\mathbb{Q}_1 \times \mathbb{P}_0$ pair

A direct consequence of the first part:

Lemma

For the subspace $G \subset \mathcal{M}_{\mathcal{P}_0} \subset \mathcal{M}_{\mathcal{P}}$, defined by

$$G := \{q \in \mathcal{M}_{\mathcal{P}_0} : [\![q]\!]_{\gamma_c} = 0 \text{ for } \gamma_c \in \mathcal{E}_c\},$$

there exists a constant β_G independent of aspect ratios such that

$$\sup_{\boldsymbol{v} \in \boldsymbol{Q}_{1,\mathcal{P}}} \frac{(\nabla \cdot \boldsymbol{v}, q)_\Omega}{|\boldsymbol{v}|_{1,\Omega}} \geq \beta_G \|q\|_{0,\Omega} \qquad \text{for all } q \in G\,.$$

The $\mathbb{Q}_1 \times \mathbb{P}_0$ pair

<u>The method:</u> Find $(\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$ such that

$$\boldsymbol{B}_s((\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}), (\boldsymbol{v}, q)) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \text{for all } (\boldsymbol{v}, q) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}} \,.$$

Here,

$$\boldsymbol{B}_s((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \boldsymbol{B}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) - \frac{1}{4}\tilde{S}(p; q)\,,$$

the stabilisation terms are

$$\tilde{S}(p, q) := \sum_{M \in \mathcal{P}_0} S_M(p, q) + \sum_{\gamma_c \in \mathcal{E}_c} S_{\gamma_c}(p, q)\,,$$

and

$$S_M(p, q) := \sum_{e \in \mathcal{E}_M} \frac{|K|}{|e|} \int_e [\![p]\!][\![q]\!] \quad \text{and} \quad S_{\gamma_c}(p, q) := \sum_{e \subset \gamma_c} \frac{\min\{|K|, |K'|\}}{|e|} \int_e [\![p]\!][\![q]\!]\,.$$

<u>Remark :</u> Without the terms S_{γ_c} the method has been proposed by L& S.

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The $\mathbb{Q}_1 \times \mathbb{P}_0$ pair

The method: Find $(\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$ such that

$$\boldsymbol{B}_s((\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}), (\boldsymbol{v}, q)) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \text{for all } (\boldsymbol{v}, q) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}} .$$

Here,

$$\boldsymbol{B}_s((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \boldsymbol{B}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) - \frac{1}{4}\tilde{S}(p; q) ,$$

the stabilisation terms are

$$\tilde{S}(p, q) := \sum_{M \in \mathcal{P}_0} S_M(p, q) + \sum_{\gamma_c \in \mathcal{E}_c} S_{\gamma_c}(p, q) ,$$

and

$$S_M(p, q) := \sum_{e \in \mathcal{E}_M} \frac{|K|}{|e|} \int_e [\![p]\!][\![q]\!] \quad \text{and} \quad S_{\gamma_c}(p, q) := \sum_{e \subset \gamma_c} \frac{\min\{|K|, |K'|\}}{|e|} \int_e [\![p]\!][\![q]\!] .$$

Remark : Without the terms S_{γ_c} the method has been proposed by L& S.

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The $\mathbb{Q}_1 \times \mathbb{P}_0$ pair

Theorem

The stabilising terms S_M and S_{γ_c} control all the unstable modes. Then, there exists a constant $\mu_s > 0$ independent of the aspect ratio ϱ, such that

$$\sup_{(\boldsymbol{v},q)\in \boldsymbol{Q}_{1,\mathcal{P}}\times\mathcal{M}_{\mathcal{P}}} \frac{\boldsymbol{B}_s((\boldsymbol{w},r)(\boldsymbol{v},q))}{|||(\boldsymbol{v},q)|||} \geq \mu_s \, |||(\boldsymbol{w},r)||| \qquad \textit{for all } (\boldsymbol{w},r) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}\,.$$

Moreover, if $p \in H^1(\Omega)$, then there exists a positive constant C such that

$$|||(\boldsymbol{u}-\boldsymbol{u}_{\mathcal{P}}, p - p_{\mathcal{P}})||| \leq (1 + C\mu_s^{-1})$$

$$\left(\inf_{(\boldsymbol{v}_{\mathcal{P}},q_{\mathcal{P}})\in \boldsymbol{Q}_{1,\mathcal{P}}\times\mathcal{M}_{\mathcal{P}}} |||(\boldsymbol{u}-\boldsymbol{v}_{\mathcal{P}}, p - q_{\mathcal{P}})||| + \sum_{K\in\mathcal{P}} h_{K,x}\|\partial_x p\|_{0,K} + h_{K,y}\|\partial_y p\|_{0K}\right)$$

where $h_{K,x}$ and $h_{K,y}$ are the diameters of $K \in \mathcal{P}$ in the x- and y-directions, respectively.

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The $\mathbb{Q}_1 \times \mathbb{P}_0$ pair

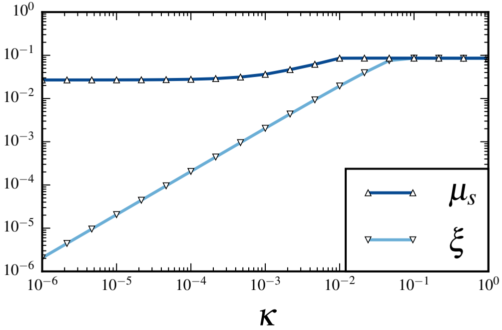

Figure 6 : Stability constants μ_s, and the LS method for a T-mesh.

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The Oseen equation

The Oseen equation : Find a pair $(\boldsymbol{u}, p)$ such that $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$, and

$$-\nu\Delta\boldsymbol{u} + \boldsymbol{b}\cdot\nabla\boldsymbol{u} + \sigma\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad , \quad \nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega\,,$$

where $\sigma, \nu > 0$ and $\nabla\cdot\boldsymbol{b} = 0$ in Ω.

Remark: We use the same finite element spaces as before. So, the stabilisation mechanisms for the pressure are identical.

The stabilised finite element method: Find $(\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$ such that:

$$\boldsymbol{B}((\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}), (\boldsymbol{v}, q)) + s_v(\boldsymbol{u}_{\mathcal{P}}, \boldsymbol{v}) - \alpha_p s_p(p, q) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}\,,$$

where

The Oseen equation

The stabilised finite element method: Find $(\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$ such that:

$$\boldsymbol{B}((\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}), (\boldsymbol{v}, q)) + s_v(\boldsymbol{u}_{\mathcal{P}}, \boldsymbol{v}) - \alpha_p s_p(p, q) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}\,,$$

where

$$\boldsymbol{B}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) := \underbrace{\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega + (\boldsymbol{b} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_\Omega + \sigma(\boldsymbol{u}, \boldsymbol{v})_\Omega}_{=:a(\boldsymbol{u},\boldsymbol{v})} - (p, \nabla \cdot \boldsymbol{v})_\Omega - (q, \nabla \cdot \boldsymbol{u})_\Omega\,,$$

The Oseen equation

The stabilised finite element method: Find $(\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$ such that:

$$\boldsymbol{B}((\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}), (\boldsymbol{v}, q)) + s_v(\boldsymbol{u}_{\mathcal{P}}, \boldsymbol{v}) - \alpha_p s_p(p, q) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}\,,$$

where

$$s_p(p, q) = \text{ exactly as before}\,,$$

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The Oseen equation

The stabilised finite element method: Find $(\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$ such that:

$$\boldsymbol{B}((\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}), (\boldsymbol{v}, q)) + s_v(\boldsymbol{u}_{\mathcal{P}}, \boldsymbol{v}) - \alpha_p s_p(p, q) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}\,,$$

where

$s_v(\boldsymbol{u}, \boldsymbol{v})$ is symmetric and needs to satisfy: Let $\|\boldsymbol{v}\|_s^2 := s_v(\boldsymbol{v}, \boldsymbol{v})$. We assume:

$$\begin{aligned} s_v(\boldsymbol{w}, \boldsymbol{v}) &\le c_s \|\boldsymbol{w}\|_s |\boldsymbol{v}|_{1,\Omega}\,, \\ s_v(\boldsymbol{v}, \boldsymbol{v}) &\ge 0\,, \\ \sum_{K \in \mathcal{P}} \gamma_K \|\kappa_K (\nabla \cdot \boldsymbol{v})\|_{0,K}^2 &\le s_v(\boldsymbol{v}, \boldsymbol{v})\,, \end{aligned}$$

for all $\boldsymbol{v}, \boldsymbol{w} \in H_0^1(\Omega)^2$.

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The Oseen equation

The stabilised finite element method: Find $(\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$ such that:

$$\boldsymbol{B}((\boldsymbol{u}_{\mathcal{P}}, p_{\mathcal{P}}), (\boldsymbol{v}, q)) + s_v(\boldsymbol{u}_{\mathcal{P}}, \boldsymbol{v}) - \alpha_p s_p(p, q) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}\,,$$

where
$s_v(\boldsymbol{u}, \boldsymbol{v})$ is symmetric and needs to satisfy: Let $\|\boldsymbol{v}\|_s^2 := s_v(\boldsymbol{v}, \boldsymbol{v})$. We assume:

$$s_v(\boldsymbol{w}, \boldsymbol{v}) \leq c_s \|\boldsymbol{w}\|_s |\boldsymbol{v}|_{1,\Omega}\,,$$
$$s_v(\boldsymbol{v}, \boldsymbol{v}) \geq 0\,,$$
$$\sum_{K \in \mathcal{P}} \gamma_K \|\kappa_K (\nabla \cdot \boldsymbol{v})\|_{0,K}^2 \leq s_v(\boldsymbol{v}, \boldsymbol{v})\,,$$

for all $\boldsymbol{v}, \boldsymbol{w} \in H_0^1(\Omega)^2$. Using these conditions, we take $\alpha_p \geq \alpha := (c_a^2 + c_s^2)^{-1}$.

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The Oseen equation

Lemma

Let s_v satisfy the previous assumptions. Let us define the mesh-dependent norm

$$|||(\boldsymbol{v},q)||| := \|\boldsymbol{v}\|_{a+s}^2 + \alpha\,\|q\|_{0,\Omega}^2 + s_p(q,q)\,.$$

Then, there exist $\mu_s > 0$, independent of the aspect ratio of the mesh, and of ν, such that:

$$\sup_{(\boldsymbol{v},q)\in \boldsymbol{Q}_{1,\mathcal{P}}\times\mathcal{M}_{\mathcal{P}}} \frac{\boldsymbol{B}_s((\boldsymbol{w},r),(\boldsymbol{v},q))}{|||(\boldsymbol{v},q)|||} \geq \mu_s\,|||(\boldsymbol{w},r)||| \qquad \text{for all } (\boldsymbol{w},r)\in \boldsymbol{Q}_{1,\mathcal{P}}\times\mathcal{M}_{\mathcal{P}}\,,$$

where $\mu_s = \beta_G^2/\left[2(1+\beta_G)(17+16\beta_G)\right]$

Moreover, error estimates in the triple norm, with constants independent of ν, can be proved.

The Oseen equation

Lemma

Let s_v satisfy the previous assumptions. Let us define the mesh-dependent norm

$$|||(\boldsymbol{v},q)||| := \|\boldsymbol{v}\|_{a+s}^2 + \alpha\,\|q\|_{0,\Omega}^2 + s_p(q,q)\,.$$

Then, there exist $\mu_s > 0$, independent of the aspect ratio of the mesh, and of ν, such that:

$$\sup_{(\boldsymbol{v},q)\in\boldsymbol{Q}_{1,\mathcal{P}}\times\mathcal{M}_{\mathcal{P}}} \frac{\boldsymbol{B}_s((\boldsymbol{w},r),(\boldsymbol{v},q))}{|||(\boldsymbol{v},q)|||} \geq \mu_s\,|||(\boldsymbol{w},r)||| \qquad \textit{for all } (\boldsymbol{w},r)\in\boldsymbol{Q}_{1,\mathcal{P}}\times\mathcal{M}_{\mathcal{P}}\,,$$

where $\mu_s = \beta_G^2/\left[2(1+\beta_G)(17+16\beta_G)\right]$

Moreover, error estimates in the triple norm, with constants independent of ν, can be proved.

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Some numerics for Oseen: some concrete choices for s_v

Defining $\kappa_D := id - \Pi_0^D$, we can define the following stabilising terms:

$$s_v^1(\boldsymbol{u}, \boldsymbol{v}) := \sum_{M \in \mathcal{P}_0} \gamma_M (\kappa_M(\nabla \cdot \boldsymbol{u}), \kappa_M(\nabla \cdot \boldsymbol{v}))_M$$

$$+ \sum_{K \in \mathcal{P}} (\kappa_K(\boldsymbol{b}_K \cdot \nabla \boldsymbol{u}), \kappa_K(\boldsymbol{b}_K \cdot \nabla \boldsymbol{v}))_K \,,$$

$$s_v^2(\boldsymbol{u}, \boldsymbol{v}) := \sum_{M \in \mathcal{P}_0} \delta_x \left(\kappa_M(\partial_x \boldsymbol{u}), \partial_x \boldsymbol{v}\right)_M + \delta_y \left(\kappa_M(\partial_y \boldsymbol{u}), \partial_y \boldsymbol{v}\right)_M \quad \text{(LPS)}\,,$$

where

Some numerics for Oseen: some concrete choices for s_v

Defining $\kappa_D := id - \Pi_0^D$, we can define the following stabilising terms:

$$
\begin{aligned}
s_v^1(\boldsymbol{u},\boldsymbol{v}) &:= \sum_{M\in\mathcal{P}_0} \gamma_M(\kappa_M(\nabla\cdot\boldsymbol{u}),\kappa_M(\nabla\cdot\boldsymbol{v}))_M \\
&\quad + \sum_{K\in\mathcal{P}} (\kappa_K(\boldsymbol{b}_K\cdot\nabla\boldsymbol{u}),\kappa_K(\boldsymbol{b}_K\cdot\nabla\boldsymbol{v}))_K\,, \\
s_v^2(\boldsymbol{u},\boldsymbol{v}) &:= \sum_{M\in\mathcal{P}_0} \delta_x\,(\kappa_M(\partial_x\boldsymbol{u}),\partial_x\boldsymbol{v})_M + \delta_y\,(\kappa_M(\partial_y\boldsymbol{u}),\partial_y\boldsymbol{v})_M \qquad (\text{LPS})\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_{K,x} &:= \nu^{-1}\|\boldsymbol{b}\|_{\infty,K}^2 h_{K,x}^2 \min\{1,\mathrm{Pe}_{\min,K}^{-1}\} \\
\delta_{K,y} &:= \nu^{-1}\|\boldsymbol{b}\|_{\infty,K}^2 h_{K,y}^2 \min\{1,\mathrm{Pe}_{\min,K}^{-1}\} \\
\mathrm{Pe}_{\min,K} &:= \nu^{-1}\min\{h_{K,x},h_{K,y}\}\|\boldsymbol{b}\|_{\infty,K}
\end{aligned}
$$

Some numerics for Oseen: some concrete choices for s_v

Defining $\kappa_D := id - \Pi_0^D$, we can define the following stabilising terms:

$$
\begin{aligned}
s_v^1(\boldsymbol{u},\boldsymbol{v}) &:= \sum_{M\in\mathcal{P}_0} \gamma_M (\kappa_M(\nabla\cdot\boldsymbol{u}),\kappa_M(\nabla\cdot\boldsymbol{v}))_M \\
&\quad + \sum_{K\in\mathcal{P}} (\kappa_K(\boldsymbol{b}_K\cdot\nabla\boldsymbol{u}),\kappa_K(\boldsymbol{b}_K\cdot\nabla\boldsymbol{v}))_K\,, \\
s_v^2(\boldsymbol{u},\boldsymbol{v}) &:= \sum_{M\in\mathcal{P}_0} \delta_x\left(\kappa_M(\partial_x\boldsymbol{u}),\partial_x\boldsymbol{v}\right)_M + \delta_y\left(\kappa_M(\partial_y\boldsymbol{u}),\partial_y\boldsymbol{v}\right)_M \quad \text{(LPS)}\,,
\end{aligned}
$$

where

$$\gamma_M := \max\{1, Pe_{\mathcal{P}_0}^{\min}\} \quad \text{where} \quad , Pe_{\mathcal{P}_0}^{\min} := \min_{M\in\mathcal{P}_0} \frac{\|\boldsymbol{b}\|_{\infty,M}}{\nu} \min\{h_{x,M}, h_{y,M}\}\,,$$

or

$$\gamma_M := 1 + ind(M) Pe_M^{\min} \quad \text{where} \quad ind(M) := 1 - \frac{\rho_M |M|}{\max_{M\in\mathcal{P}_0}|M|}\,.$$

Some numerics for Oseen: some concrete choices for s_v

The mesh:

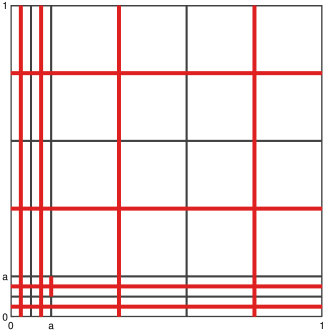

Figure 7 : A Shishkin mesh with parameter $\lambda = \min\{\frac{1}{2},\ 2\,\nu \ln N\}$ ($\nu = 1/32$), with $N = 8$ intervals.

The solution:

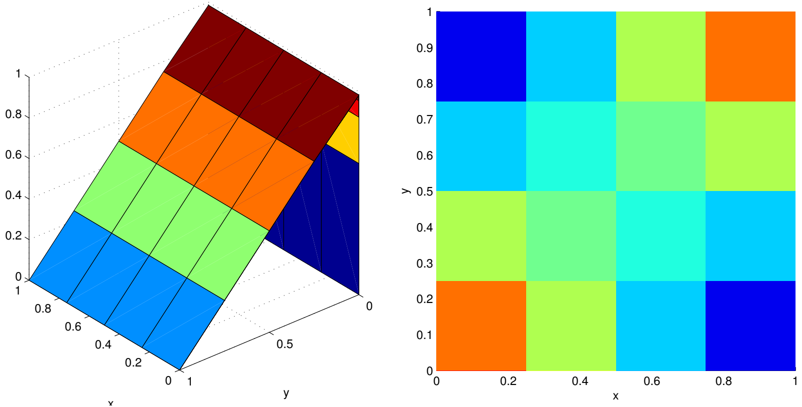

Figure 8 : Nodal interpolation of $\boldsymbol{u}_1$ (left) and L^2 projection of p (right) for $\nu = 10^{-6}$.

Some numerics for Oseen: some concrete choices for s_v

Some error results ($\nu = 10^{-6}$): We define, for a given partition $\mathcal{P}$, the relative errors

$$E_p^{\text{rel}} := \frac{\|p - p_{\mathcal{P}}\|_{0,\Omega}}{\|p - \Pi p\|_{0,\Omega}} \quad \text{and} \quad E_{\boldsymbol{u}}^{\text{rel}} := \frac{|\boldsymbol{u} - \boldsymbol{u}_{\mathcal{P}}|_{1,\Omega}}{|\boldsymbol{u} - I_{\mathcal{P}}\boldsymbol{u}|_{1,\Omega}}\,.$$

Table 1 : Here, $N = 8$, $\lambda = 0.01$, and $\boldsymbol{Q}_{1,\mathcal{P}} \times G$.

s^v	γ_M	E_p^{rel}	$E_{\boldsymbol{u}}^{\text{rel}}$
s_v^1	First	25.02	1.0
s_v^1	Second	25.31	1.0
LPS	-	1.40	1.0002

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Some numerics for Oseen: some concrete choices for s_v

Some error results ($\nu = 10^{-6}$): We define, for a given partition $\mathcal{P}$, the relative errors

$$E_p^{\text{rel}} := \frac{\|p - p_{\mathcal{P}}\|_{0,\Omega}}{\|p - \Pi p\|_{0,\Omega}} \quad \text{and} \quad E_{\boldsymbol{u}}^{\text{rel}} := \frac{|\boldsymbol{u} - \boldsymbol{u}_{\mathcal{P}}|_{1,\Omega}}{|\boldsymbol{u} - I_{\mathcal{P}}\boldsymbol{u}|_{1,\Omega}}.$$

Table 1 : Here, $N = 8$, $\lambda = 0.0001$, and $\boldsymbol{Q}_{1,\mathcal{P}} \times G$.

s^v	γ_M	E_p^{rel}	$E_{\boldsymbol{u}}^{\text{rel}}$
s_v^1	First	1.06	1.0
s_v^1	Second	1.06	1.0
LPS	-	1.22	1.0219

Some numerics for Oseen: some concrete choices for s_v

Some error results ($\nu = 10^{-6}$): We define, for a given partition $\mathcal{P}$, the relative errors

$$E_p^{\text{rel}} := \frac{\|p - p_{\mathcal{P}}\|_{0,\Omega}}{\|p - \Pi p\|_{0,\Omega}} \quad \text{and} \quad E_{\boldsymbol{u}}^{\text{rel}} := \frac{|\boldsymbol{u} - \boldsymbol{u}_{\mathcal{P}}|_{1,\Omega}}{|\boldsymbol{u} - I_{\mathcal{P}}\boldsymbol{u}|_{1,\Omega}} \,.$$

Table 1 : Here, $N = 8$, $\lambda = 0.01$, and $\boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$, and $\alpha_p = 1$.

s^v	γ_M	E_p^{rel}	$E_{\boldsymbol{u}}^{\text{rel}}$
s_v^1	First	47.79	1.0
s_v^1	Second	48.63	1.0
LPS	-	7.47	1.0002

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Some numerics for Oseen: some concrete choices for s_v

Some error results ($\nu = 10^{-6}$): We define, for a given partition $\mathcal{P}$, the relative errors

$$E_p^{\rm rel} := \frac{\|p - p_{\mathcal{P}}\|_{0,\Omega}}{\|p - \Pi p\|_{0,\Omega}} \quad \text{and} \quad E_{\boldsymbol{u}}^{\rm rel} := \frac{|\boldsymbol{u} - \boldsymbol{u}_{\mathcal{P}}|_{1,\Omega}}{|\boldsymbol{u} - I_{\mathcal{P}}\boldsymbol{u}|_{1,\Omega}}\,.$$

Table 1 : Here, $N = 8$, $\lambda = 0.0001$, and $\boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$, and $\alpha_p = 1$.

s^v	γ_M	$E_p^{\rm rel}$	$E_{\boldsymbol{u}}^{\rm rel}$
s_v^1	First	3.06	1.0
s_v^1	Second	2.78	1.0
LPS	-	6.73	1.0152

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Some numerics for Oseen: some concrete choices for s_v

Shaprness of the layers with varying λ.

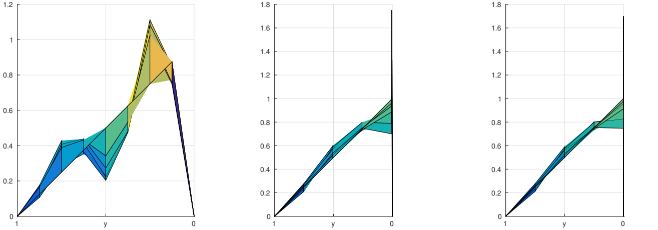
s_v, γ_M	$\lambda = 0.5000$	$\lambda = 1.0000 \cdot 10^{-2}$	$\lambda = 1.0000 \cdot 10^{-4}$
LPS			

Figure 9 : Meshes: $N = 8$, λ (left to right). Using $\boldsymbol{Q}_{1,\mathcal{P}} \times G$.

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Shaprness of the layers with varying λ.

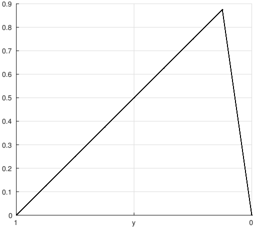
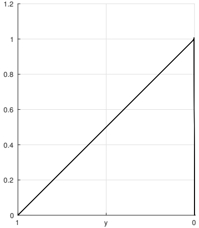
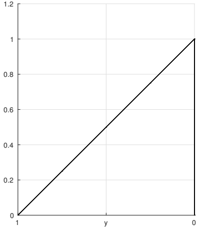
s_v, γ_M	$\lambda = 0.5000$	$\lambda = 1.0000 \cdot 10^{-2}$	$\lambda = 1.0000 \cdot 10^{-4}$
s_v^1 + First			

Figure 9 : Meshes: $N = 8$, λ (left to right). Using $\boldsymbol{Q}_{1,\mathcal{P}} \times G$.

Shaprness of the layers with varying λ.

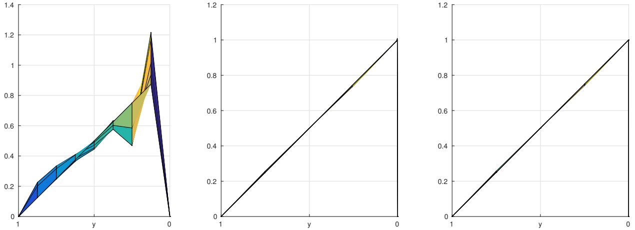
s_v, γ_M	$\lambda = 0.5000$	$\lambda = 1.0000 \cdot 10^{-2}$	$\lambda = 1.0000 \cdot 10^{-4}$
s_v^1 + Second			

Figure 9 : Meshes: $N = 8$, λ (left to right). Using $\boldsymbol{Q}_{1,\mathcal{P}} \times G$.

Some numerics for Oseen: some concrete choices for s_v

Shaprness of the layers with varying λ.

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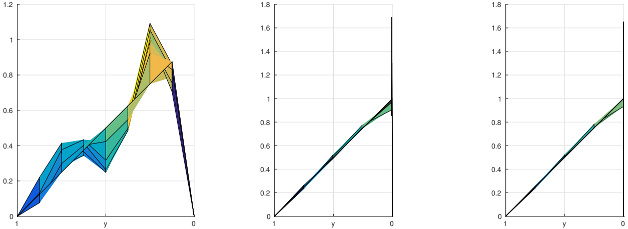
LPS			

Figure 10 : Meshes: $N = 8$, λ (left to right). Using $\boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$, $\alpha_p = 1$.

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Some numerics for Oseen: some concrete choices for s_v

Shaprness of the layers with varying λ.

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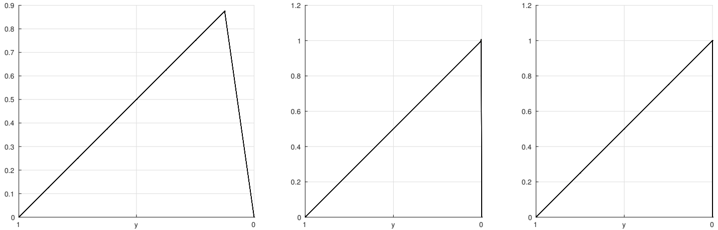

Figure 10 : Meshes: $N = 8$, λ (left to right). Using $\boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$, $\alpha_p = 1$.

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Shaprness of the layers with varying λ.

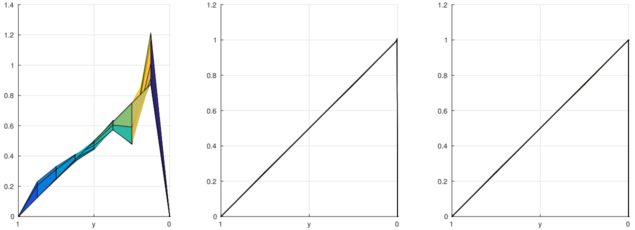
s_v, γ_M	$\lambda = 0.5000$	$\lambda = 1.0000 \cdot 10^{-2}$	$\lambda = 1.0000 \cdot 10^{-4}$
S_v^1 + Second			

Figure 10 : Meshes: $N = 8$, λ (left to right). Using $\boldsymbol{Q}_{1,\mathcal{P}} \times \mathcal{M}_{\mathcal{P}}$, $\alpha_p = 1$.

Conclusions and perspectives

1. Identification of the minimal number of spurious pressure modes on anisotropic meshes.
2. A new family of inf-sup stable finite element spaces. These enjoy the same approximation properties of the original one.
3. A stabilised variant penalising these modes in the formulation: stability and optimal convergence.
4. Extension to the (optimal) $\mathbb{Q}_1^2 \times \mathbb{P}_0$ pair, and Oseen.

Perspectives and open questions:

- Adaptivity?
- Triangles?
- Continuous pressures?

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