The number of maximal sum-free subsets of integers

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Abstract

Cameron and Erdős [6] raised the question of how many *maximal* sum-free sets there are in $\{1,\dots,n\}$, giving a lower bound of $2^{\lfloor n/4 \rfloor}$. In this paper we prove that there are in fact at most $2^{(1/4+o(1))n}$ maximal sum-free sets in $\{1,\dots,n\}$. Our proof makes use of container and removal lemmas of Green [8, 9] as well as a result of Deshouillers, Freiman, Sós and Temkin [7] on the structure of sum-free sets.

1 Introduction

A fundamental notion in combinatorial number theory is that of a sum-free set: A set S of integers is *sum-free* if $x+y \notin S$ for every $x, y \in S$ (note x and y are not necessarily distinct here). The topic of sum-free sets of integers has a long history. Indeed, in 1916 Schur [19] proved that, if n is sufficiently large, then any r-colouring of $[n] := \{1,\dots,n\}$ yields a monochromatic triple x, y, z such that $x+y=z$.

Note that both the set of odd numbers in $[n]$ and the set $\{\lfloor n/2 \rfloor + 1,\dots,n\}$ are maximal sum-free sets. (A sum-free subset of $[n]$ is *maximal* if it is not properly contained in another sum-free subset of $[n]$.) By considering all possible subsets of one of these maximal sum-free sets, we see that $[n]$ contains at least $2^{\lceil n/2 \rceil}$ sum-free sets. Cameron and Erdős [5] conjectured that in fact $[n]$ contains only $O(2^{n/2})$ sum-free sets. The conjecture was proven independently by Green [8] and Sapozhenko [16]. Recently, a refinement of the Cameron–Erdős conjecture was proven in [1], giving an upper bound on the number of sum-free sets in $[n]$ of size m (for each $1 \le m \le \lceil n/2 \rceil$).

Let $f(n)$ denote the number of sum-free subsets of $[n]$ and $f_{\max}(n)$ denote the number of maximal sum-free subsets of $[n]$. Recall that the sum-free subsets of $[n]$ described above lie in

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just two maximal sum-free sets. This led Cameron and Erdős [6] to ask whether the number of maximal sum-free subsets of $[n]$ is "substantially smaller" than the total number of sum-free sets. In particular, they asked whether $f_{\max}(n) = o(f(n))$ or even $f_{\max}(n) \le f(n)/2^{\varepsilon n}$ for some constant $\varepsilon > 0$. Łuczak and Schoen [14] answered this question, showing that $f_{\max}(n) \le 2^{n/2-2^{-28}n}$ for sufficiently large n. More recently, Wolfovitz [20] proved that $f_{\max}(n) \le 2^{3n/8+o(n)}$.

In the other direction, Cameron and Erdős [6] observed that $f_{\max}(n) \ge 2^{\lfloor n/4 \rfloor}$. Indeed, let $m = n$ or $m = n-1$, whichever is even. Let S consist of m together with precisely one number from each pair $\{x, m-x\}$ for odd $x < m/2$. Then S is sum-free. Moreover, although S may not be maximal, no further odd numbers less than m can be added, so distinct S lie in distinct maximal sum-free subsets of $[n]$.

We prove that this lower bound is in fact, 'asymptotically', the correct bound on $f_{\max}(n)$.

Theorem 1.1. *There are at most $2^{(1/4+o(1))n}$ maximal sum-free sets in $[n]$. That is,*

$$f_{\max}(n) = 2^{(1/4+o(1))n}.$$

The proof of Theorem 1.1 makes use of 'container' and 'removal' lemmas of Green [8, 9] as well as a result of Deshouillers, Freiman, Sós and Temkin [7] on the structure of sum-free sets (see Section 2 for an overview of the proof).

Next we provide another collection of maximal sum-free sets in $[n]$. Suppose that $4|n$ and set $I_1 := \{n/2+1, \dots, 3n/4\}$ and $I_2 := \{3n/4+1, \dots, n\}$. First choose the element $n/4$ and a set $S \subseteq I_2$. Then for every $x \in I_2 \setminus S$, choose $x - n/4 \in I_1$. The resulting set is sum-free but may not be maximal. However, no further element in I_2 can be added, thus distinct S lie in distinct maximal sum-free sets in $[n]$. There are $2^{|I_2|} = 2^{n/4}$ ways to choose S.

It would be of interest to establish whether $f_{\max}(n) = O(2^{n/4})$.

Question 1.2. Does $f_{\max}(n) = O(2^{n/4})$?

A solution to the question is a current work in progress. In a forthcoming paper [2] we consider the analogous problem for maximal sum-free sets in abelian groups.

Notation: Given a set $A \subseteq [n]$, denote by $f_{\max}(A)$ the number of maximal sum-free subsets of $[n]$ that lie in A and by $\min(A)$ the minimum element of A. Let $1 \le p < q \le n$ be integers, denote $[p,q] := \{p, p+1, \dots, q\}$. Denote by E the set of all even numbers in $[n]$ and by O the set of all odd numbers in $[n]$. A triple $x, y, z \in [n]$ is called a *Schur triple* if $x + y = z$ (here $x = y$ is allowed).

Throughout, all graphs considered are simple unless stated otherwise. We say that a graph G is a *graph possibly with loops* if G can be obtained from a simple graph by adding at most one loop at each vertex. Given a vertex x in G, we write $\deg_G(x)$ for the *degree of x in G*. Note that a loop at x contributes two to the degree of x. We write $\delta(G)$ for the *minimum degree of* G and $\Delta(G)$ for the *maximum degree of* G. Given a graph G, denote by $\mathrm{MIS}(G)$ the number of maximal independent sets in G. Given $T \subseteq V(G)$, denote by $\Gamma(T)$ the external neighbourhood of T, i.e. $\Gamma(T) := \{v \in V(G) \setminus T : \exists u \in T, uv \in E(G)\}$. Denote by $G[T]$ the induced subgraph of G on the vertex set T and let $G \setminus T$ denote the induced subgraph of G on the vertex set $V(G) \setminus T$. Denote by $E(T)$ the set of edges in G spanned by T and by $E(T, V(G) \setminus T)$ the set of edges in G with exactly one vertex in T.

2 Overview of the proof and preliminary results

2.1 Proof overview

We prove Theorem 1.1 in Section 3. A key tool in the proof is the following container lemma of Green [8] for sum-free sets. The first container-type result in the area (for counting sum-free subsets of $\mathbb{Z}_p$) was given by Green and Ruzsa [10].

Lemma 2.1 (Proposition 6 in [8]). *There exists a family $\mathcal{F}$ of subsets of $[n]$ with the following properties.*

(i) Every member of $\mathcal{F}$ has at most $o(n^2)$ Schur triples.

(ii) If $S \subseteq [n]$ is sum-free, then S is contained in some member of $\mathcal{F}$.

(iii) $|\mathcal{F}| = 2^{o(n)}$.

(iv) Every member of $\mathcal{F}$ has size at most $(1/2 + o(1))n$.

We refer to the elements of $\mathcal{F}$ from Lemma 2.1 as *containers*. In [8], condition (iv) was not stated explicitly. However, it follows immediately from (i) by, for example, applying Theorem 2.2 and Lemma 2.3 below. Lemma 2.1 can also be derived from a general theorem of Balogh, Morris and Samotij [3], and independently Saxton and Thomason [18] with better bounds in (i) and (iii).

Note that conditions (ii) and (iii) in Lemma 2.1 imply that, to prove Theorem 1.1, it suffices to show that every member of $\mathcal{F}$ contains at most $2^{n/4+o(n)}$ maximal sum-free subsets of $[n]$. For this purpose, we need to get a handle on the structure of the containers; this is made precise in Lemma 2.4 below. The following theorem of Deshouillers, Freiman, Sós and Temkin [7] provides a structural characterisation of the sum-free sets in $[n]$.

Theorem 2.2. *Every sum-free set S in $[n]$ satisfies at least one of the following conditions:*

(i) $|S| \leq 2n/5 + 1$;

(ii) S consists of odd numbers;

(iii) $|S| \leq \min(S)$.

We also need the following removal lemma of Green [9] for sum-free sets. (A simpler proof of Lemma 2.3 was later given by Král', Serra and Vena [13].)

Lemma 2.3 (Corollary 1.6 in [9]). *Suppose that $A \subseteq [n]$ is a set containing $o(n^2)$ Schur triples. Then, there exist B and C such that $A = B \cup C$ where B is sum-free and $|C| = o(n)$.*

Together, Theorem 2.2 and Lemma 2.3 yield the following structural result on containers of size close to $n/2$.

Lemma 2.4. *If $A \subseteq [n]$ has $o(n^2)$ Schur triples and $|A| = (\frac{1}{2} - \gamma)n$ with $\gamma = \gamma(n) \leq 1/11$, then one of the following conditions holds.*

(a) All but $o(n)$ elements of A are contained in the interval $[(1/2 - \gamma)n, n]$.

(b) Almost all elements of A are odd, i.e. $|A \setminus O| = o(n)$.

Proof. Apply Lemma 2.3 to A; we have $A = B \cup C$ with B sum-free and $|C| = o(n)$. Apply Theorem 2.2 to B. Alternative (i) is impossible, since $|B| \geq (1 - o(1))|A| > 2n/5 + 1$. If alternative (ii) occurs, then we have $|A \setminus O| \leq |C| = o(n)$. If alternative (iii) occurs, then $\min(B) \geq |B| \geq (1/2 - \gamma - o(1))n$. So all but except $o(n)$ elements of A are contained in $[(1/2 - \gamma)n, n]$. □

We remark that Lemma 2.4 was already essentially proven in [8] (without applying Lemma 2.3). Note that γ could be negative in Lemma 2.4. The upper bound 1/11 on γ here can be relaxed to any constant smaller than 1/10 (but not to a constant bigger than 1/10). Roughly speaking, Lemma 2.4 implies that every container $A \in \mathcal{F}$ is such that (a) most elements of A lie in $[n/2, n]$, (b) most elements of A are odd or (c) $|A|$ is significantly smaller than $n/2$. Thus, the proof of Theorem 1.1 splits into three cases depending on the structure of our container. In each case, we give an upper bound on the number of maximal sum-free sets in a container by counting the number of maximal independent sets in various auxiliary graphs. (Similar techniques were used in [20], and in the graph setting in [4].) In the following subsection we collect together a number of results that are useful for this.

2.2 Maximal independent sets in graphs

Moon and Moser [15] showed that for any graph G, $\mathrm{MIS}(G) \leq 3^{|G|/3}$. We will need a looped version of this statement. Since any vertex with a loop cannot be in an independent set, the following statement is an immediate consequence of Moon and Moser's result.

Proposition 2.5. *Let G be a graph possibly with loops. Then*

$$\mathrm{MIS}(G) \leq 3^{|G|/3}.$$

When a graph is triangle-free, the bound in Proposition 2.5 can be improved significantly. A result of Hujter and Tuza [11] states that for any triangle-free graph G,

$$\mathrm{MIS}(G) \leq 2^{|G|/2}. \tag{1}$$

The following lemma is a slight modification of this result for graphs with 'few' triangles.

Lemma 2.6. *Let G be a graph possibly with loops. If there exists a set T such that $G \setminus T$ is triangle-free, then*

$$\mathrm{MIS}(G) \leq 2^{|G|/2 + |T|/2}.$$

Proof. Every maximal independent set in G can be obtained in the following two steps:

(1) Choose an independent set $S \subseteq T$.

(2) Extend S in $V(G) \setminus T$, i.e. choose a set $R \subseteq V(G) \setminus T$ such that $R \cup S$ is a maximal independent set in G.

Note that although every maximal independent set in G can be obtained in this way, it is not necessarily the case that given an arbitrary independent set $S \subseteq T$, there exists a set $R \subseteq V(G) \setminus T$ such that $R \cup S$ is a maximal independent set in G. Notice that if $R \cup S$ is maximal, R is also a maximal independent set in $G \setminus \{T \cup \Gamma(S)\}$. The number of choices for S in (1) is at most $2^{|T|}$. Since $G \setminus \{T \cup \Gamma(S)\}$ is triangle-free, by the Hujter–Tuza bound, the number of extensions in (2) is at most $2^{(|G|-|T|)/2}$. Thus, we have $\mathrm{MIS}(G) \le 2^{|T|} \cdot 2^{(|G|-|T|)/2} = 2^{|G|/2+|T|/2}$. □

The following lemma gives an improvement on Proposition 2.5 for graphs that are 'not too sparse and almost regular'. The proof uses an elegant and simple idea of Sapozhenko [17], see [12] for a closely-related result.

Lemma 2.7. *Let $k \ge 1$ and let G be a graph on n vertices possibly with loops. Suppose that $\Delta(G) \le k\delta(G)$ where $\delta(G) \ge f(n)$ for some real valued function f with $f(n) \to \infty$ as $n \to \infty$. Then*

$$\mathrm{MIS}(G) \le 3^{\left(\frac{k}{k+1}\right)\frac{n}{3}+o(n)}.$$

Proof. Fix a maximal independent set I in G and set $b := \delta(G)^{1/2}$. We will repeat the following process as many times as possible. Let $V_1 := V(G)$. At the i-th step, for $i \ge 1$, choose $v_i \in V_i \cap I$ such that $\deg_{G[V_i]}(v_i) \ge b$ and set $V_{i+1} := V_i \setminus (\{v_i\} \cup \Gamma(v_i))$. This process is repeated $j \le n/b$ times. Let $U := V_{j+1}$ be the resulting set. Define $Z := \{v \in U : \deg_{G[U]}(v) < b\}$. Notice that $\deg_{G[U]}(v) < b$ for all $v \in I \cap U$, hence $I \cap U \subseteq Z$. We have

$$\delta(G) \cdot |Z| \le \sum_{v \in Z} \deg(v) = 2|E(Z)| + |E(Z, V \setminus Z)| \le b|Z| + \Delta(G) \cdot (n - |Z|).$$

Hence,

$$|Z| \le \frac{\Delta(G) \cdot n}{\delta(G) + \Delta(G) - b} \le \frac{k}{k+1}n + \frac{2n}{b}. \tag{2}$$

By construction of U, no vertex in $I \setminus U$ has a neighbour in U. So as $Z \subseteq U$, no vertex in Z is adjacent to $I \setminus U$. Together with the fact that I is maximal, this implies that $I \cap U$ is a maximal independent set in $G[Z]$. By the above process, every maximal independent set I in G is determined by a set $I \setminus U$ of at most n/b vertices and a maximal independent set in $G[Z]$. Note that $n/b = o(n)$. Thus, Proposition 2.5 and (2) imply that

$$\mathrm{MIS}(G) \le \sum_{0 \le i \le n/b} \binom{n}{i} 3^{\left(\frac{k}{k+1}\right)\frac{n}{3}+\frac{2n}{3b}} \le 3^{\left(\frac{k}{k+1}\right)\frac{n}{3}+o(n)}. \tag{3}$$

□

Note that one could relax the minimum degree condition in Lemma 2.7 to (for example) a large constant, at the expense of a worse upper bound on $\mathrm{MIS}(G)$. However, Lemma 2.7 in its current form suffices for our applications.

3 Proof of Theorem 1.1

Let $\mathcal{F}$ be the family of containers obtained from Lemma 2.1. Recall that given a set $A \subseteq [n]$, $f_{\max}(A)$ denotes the number of maximal sum-free subsets of $[n]$ that lie in A. Since every sum-free subset of $[n]$ is contained in some member of $\mathcal{F}$ and $|\mathcal{F}| = 2^{o(n)}$, it suffices to show that $f_{\max}(A) \leq 2^{(1/4+o(1))n}$ for every container $A \in \mathcal{F}$.

Lemmas 2.1 and 2.4 imply that every container $A \in \mathcal{F}$ satisfies at least one of the following conditions:

(a) $|A| \leq (1/2 - 1/11)n \leq 0.45n$

or one of the following holds for some $-o(1) \leq \gamma = \gamma(n) \leq 1/11$:

(b) $|A| = \left(\frac{1}{2} - \gamma\right) n$ and $|A \cap [(1/2 - \gamma)n]| = o(n)$;

(c) $|A| = \left(\frac{1}{2} - \gamma\right) n$ and $|A \setminus O| = o(n)$.

We deal with each of the three cases separately.

For any subsets $B, S \subseteq [n]$, let $L_S[B]$ be the *link graph of S on B* defined as follows. The vertex set of $L_S[B]$ is B. The edge set of $L_S[B]$ consists of the following two types of edges:

(i) Two vertices x and y are adjacent if there exists an element $z \in S$ such that $\{x, y, z\}$ forms a Schur triple;

(ii) There is a loop at a vertex x if $\{x, x, z\}$ forms a Schur triple for some $z \in S$ or if $\{x, z, z'\}$ forms a Schur triple for some $z, z' \in S$.

The following simple result will be applied in all three cases of our proof.

Lemma 3.1. *Suppose that B, S are both sum-free subsets of $[n]$. If $I \subseteq B$ is such that $S \cup I$ is a maximal sum-free subset of $[n]$, then I is a maximal independent set in $G := L_S[B]$.*

Proof. First notice that I is an independent set in G, since otherwise $S \cup I$ is not sum-free. Suppose to the contrary that there exists a vertex $v \notin I$ such that $I' := I \cup \{v\}$ is still an independent set in G. Then since $I' \subseteq B$ is sum-free, the definition of G implies that $S \cup I'$ is a sum-free set in $[n]$ containing $S \cup I$, a contradiction to the maximality of $S \cup I$. $\square$

3.1 Small containers

The following lemma deals with containers of 'small' size.

Lemma 3.2. *If $A \in \mathcal{F}$ has size at most $0.45n$, then $f_{\max}(A) = o(2^{n/4})$.*

Proof. Lemma 2.1 (i) implies that we can apply Lemma 2.3 to A to obtain that $A = B \cup C$ where B is sum-free and $|C| = o(n)$. Notice crucially that every maximal sum-free subset of $[n]$ in A can be built in the following two steps:

(1) Choose a sum-free set S in C;

(2) Extend S in B to a maximal one.

(As in Lemma 2.6, note that it is not necessarily the case that given an arbitrary sum-free set $S \subseteq C$, there exists a set $R \subseteq B$ such that $R \cup S$ is a maximal sum-free set in $[n]$.)

The number of choices for S is at most $2^{|C|} = 2^{o(n)}$. For a fixed S, denote by $N(S, B)$ the number of extensions of S in B in Step (2). It suffices to show that for any given sum-free set

$S \subseteq C$, $N(S,B) \leq 2^{0.249n}$. Let $G := L_S[B]$ be the link graph of S on B. Since $|A| \leq 0.45n$ and S and B are sum-free, Lemma 3.1 and Proposition 2.5 imply that

$$N(S,B) \leq \mathrm{MIS}(G) \leq 3^{|B|/3} \leq 3^{|A|/3} \leq 3^{0.45n/3} \ll 2^{0.249n}.$$

□

3.2 Large containers

We now turn our attention to containers of relatively large size.

Lemma 3.3. *Let* $-o(1) \leq \gamma = \gamma(n) \leq 1/11$. *If* $A \subseteq [n]$ *has* $o(n^2)$ *Schur triples,* $|A| = \left(\frac{1}{2} - \gamma\right) n$ *and* $|A \cap [(1/2-\gamma)n]| = o(n)$, *then*

$$f_{\max}(A) \leq 2^{(1/4+o(1))n}.$$

Proof. Let $A \in \mathcal{F}$ be as in the statement of the lemma. Let $A_1 := A \cap [\lfloor n/2 \rfloor]$ and $A_2 := A \setminus A_1$. Since $|A \cap [(1/2-\gamma)n]| = o(n)$, we have that $|A_1| \leq (\gamma + o(1))n$. Every maximal sum-free subset of $[n]$ in A can be built from choosing a sum-free set $S \subseteq A_1$ and extending S in A_2. The number of choices for S is at most $2^{|A_1|}$.

Let $G := L_S[A_2]$ be the link graph of S on vertex set A_2. Since S and A_2 are sum-free, Lemma 3.1 implies that $N(S, A_2) \leq \mathrm{MIS}(G)$. Notice that G is triangle-free. Indeed, suppose to the contrary that $z > y > x > n/2$ form a triangle in G. Then there exists $a, b, c \in S$ such that $z - y = a, y - x = b$ and $z - x = c$, which implies $a + b = c$ with $a, b, c \in S$. This is a contradiction to S being sum-free. Thus by (1) we have $N(S, A_2) \leq \mathrm{MIS}(G) \leq 2^{|A_2|/2}$. Then we have

$$f_{\max}(A) \leq 2^{|A_1|+|A_2|/2} = 2^{|A_1|+((1/2-\gamma)n-|A_1|)/2} = 2^{n/4+(|A_1|-\gamma n)/2} \leq 2^{n/4+o(n)},$$

where the last inequality follows since $|A_1| \leq (\gamma + o(1))n$. □

Lemma 3.4. *If* $A \in \mathcal{F}$ *such that* $|A \setminus O| = o(n)$, *then*

$$f_{\max}(A) \leq 2^{(1/4+o(1))n}.$$

Proof. Let $A \in \mathcal{F}$ be as in the statement of the lemma. Notice that if $S \subseteq T \subseteq [n]$ then $f_{\max}(S) \leq f_{\max}(T)$. Using this fact, we may assume that $A = O \cup C$ with $C \subseteq E$ and $|C| = o(n)$. Similarly to before, every maximal sum-free subset of $[n]$ in A can be built from choosing a sum-free set $S \subseteq C$ (at most $2^{|C|} = 2^{o(n)}$ choices) and extending S in O to a maximal one. Fix an arbitrary sum-free set S in C and let $G := L_S[O]$ be the link graph of S on vertex set O. Since O is sum-free, by Lemma 3.1 we have that $N(S, O) \leq \mathrm{MIS}(G)$. It suffices to show that $\mathrm{MIS}(G) \leq 2^{n/4+o(n)}$. We will achieve this in two cases depending on the size of S.

Case 1: $|S| \geq n^{1/4}$.

In this case, we will show that G is 'not too sparse and almost regular'. Then we apply Lemma 2.7.

We first show that $\delta(G) \geq |S|/2$ and $\Delta(G) \leq 2|S|+2$, thus $\Delta(G) \leq 6\delta(G)$. Let x be any vertex in O. If $s \in S$ such that $s < \max\{x, n-x\}$ then at least one of $x-s$ and $x+s$ is adjacent to x in G. If $s \in S$ such that $s \geq \max\{x, n-x\}$ then $s-x$ is adjacent to x in G. By considering all $s \in S$ this implies that $\deg_G(x) \geq |S|/2$ (we divide by 2 here as an edge xy may arise from two different elements of S). For the upper bound consider $x \in O$. If $xy \in E(G)$ then $y = x+s$, $x-s$ or $s-x$ for some $s \in S$ and only two of these terms are positive. Further, there may be a loop at x in G (contributing 2 to the degree of x in G). Thus, $\deg_G(x) \leq 2|S|+2$, as desired.

Since $\delta(G) \geq |S|/2 \geq n^{1/4}/2$ we can apply Lemma 2.7 to G with $k=6$. Hence,

$$\mathrm{MIS}(G) \leq 3^{\left(\frac{6}{7}\right)\frac{n/2}{3}+o(n)} \ll 2^{0.24n+o(n)} = o(2^{n/4}).$$

Case 2: $|S| \leq n^{1/4}$.

In this case, it suffices to show that G has very few, $o(n)$, triangles, since then by applying Lemma 2.6 with T being the vertex set of all triangles in G, we have $|T| = o(n)$ and then $\mathrm{MIS}(G) \leq 2^{n/4+o(n)}$. Recall that for each edge xy in G, at least one of the evens $x+y$ and $|x-y|$ is in S. We call xy a BLUE edge if $|x-y|$ is in S and a RED edge if $|x-y| \notin S$ and $x+y \in S$.

Claim 3.5. *Each triangle in G contains either 0 or 2 BLUE edges.*

Proof. Let xyz be a triangle in G with $x<y<z$. Suppose that xyz has only one BLUE edge xz. Then $s_1 := z-x, s_2 := x+y$ and $s_3 := y+z$ are elements of S and $s_1+s_2 = s_3$, a contradiction to S being sum-free. All other cases, including when all the edges are BLUE, are similar, we omit the proof here. □

Consider an arbitrary triple $\{s_1, s_2, s_3\}$ in S (where s_1, s_2 and s_3 are not necessarily distinct). We say that $\{s_1, s_2, s_3\}$ *forces a triangle $\mathcal{T}$ in G* if the vertex set $\{x, y, z\}$ of $\mathcal{T}$ is such that s_1, x, y; s_2, y, z and; s_3, x, z form Schur triples. Note that by definition of G, every triangle in G is forced by some triple in S.

Fix an arbitrary triple $\{s_1, s_2, s_3\}$ in S. We will show that $\{s_1, s_2, s_3\}$ forces at most 24 triangles in G. This then implies that G has at most $24|S|^3 = o(n)$ triangles as desired.

By Claim 3.5, a triangle xyz with $x<y<z$ can only be one of the following four types: (1) all edges are RED; (2) xy is the only RED edge; (3) yz is the only RED edge; (4) xz is the only RED edge.

It suffices to show that $\{s_1, s_2, s_3\}$ can force at most 6 triangles of each type. We show it only for Type (1), the other types are similar. Suppose that xyz is a Type (1) triangle forced by $\{s_1, s_2, s_3\}$. Set $M := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Then $\mathbf{u} = (x, y, z)^T$ is a solution to $M \cdot \mathbf{u} = \mathbf{s}$ for some $\mathbf{s}$ whose entries are precisely the elements of $\{s_1, s_2, s_3\}$.

Since $\det(M) = 2 \neq 0$, if a solution $\mathbf{u}$ exists to $M \cdot \mathbf{u} = \mathbf{s}$, it should be unique. The number of choices for $\mathbf{s}$, for fixed $\{s_1, s_2, s_3\}$, is $3! = 6$. Thus in total there are at most 6 triangles of Type (1) forced by $\{s_1, s_2, s_3\}$.

This completes the proof of Lemma 3.4. □

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