More on perfect matchings in hypergraphs

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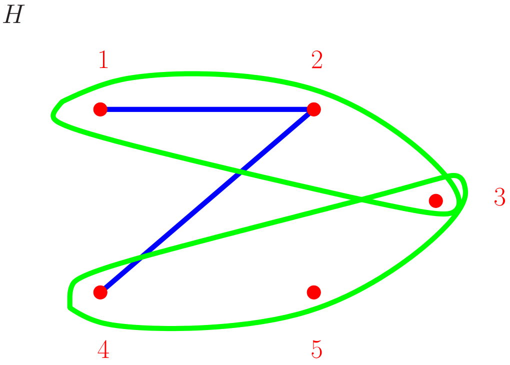
Joint work with Yi Zhao (Georgia State)

A hypergraph H is a set of vertices $V(H)$ together with a collection $E(H)$ of subsets of $V(H)$ (known as edges).

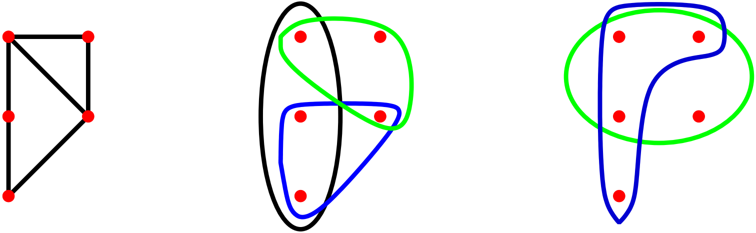
For example, consider the hypergraph H with

- $V(H) = \{1, 2, 3, 4, 5\}$;
- $E(H) = \{\{1, 2\}, \{1, 2, 3\}, \{2, 4\}, \{3, 4, 5\}\}$.

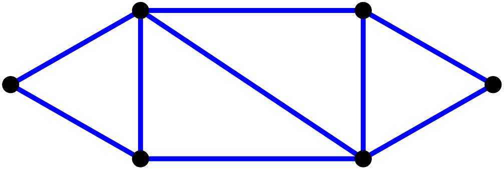


A k-uniform hypergraph H is hypergraph whose edges contain *precisely* k vertices.

- 2-uniform hypergraphs are graphs.



A matching in a hypergraph H is a collection of vertex-disjoint edges.
A perfect matching is a matching covering *all* the vertices of H.



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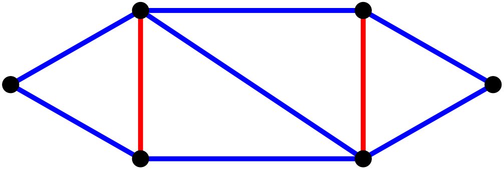

matching

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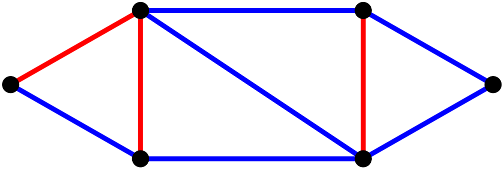

not a matching

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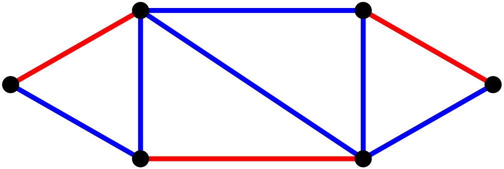


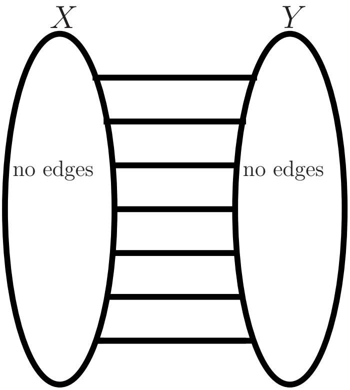
perfect matching

Theorem (Hall's Marriage theorem)

G bipartite graph with equal size vertex classes X, Y

$$G \text{ has perfect matching } \iff \forall\, S \subseteq X,\ |N(S)| \geq |S|$$

($N(S)$ = set of vertices that receive at least one edge from S)



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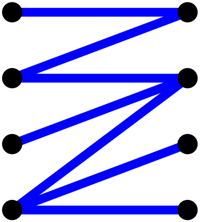

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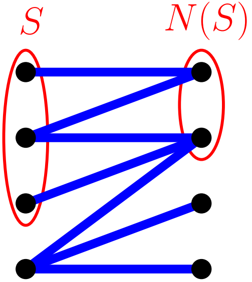
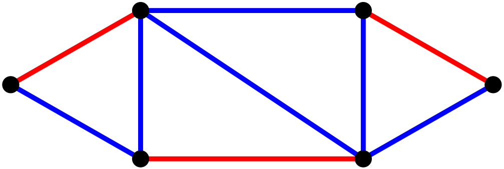


no perfect matching

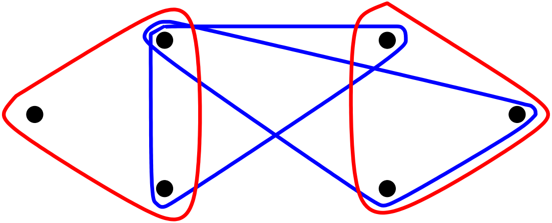
Characterising graphs with perfect matchings

- Tutte's Theorem characterises all those graphs with perfect matchings.



Perfect matchings in k-uniform hypergraphs

- for $k \geq 3$ decision problem NP-complete (Garey, Johnson '79)
- Natural to look for simple sufficient conditions



minimum ℓ-degree conditions

- H k-uniform hypergraph, $1 \leq \ell < k$
- $d_H(v_1, \dots, v_\ell) = \#$ edges containing $v_1, \dots, v_\ell$
- minimum ℓ-degree $\delta_\ell(H)$ = minimum over all $d_H(v_1, \dots, v_\ell)$
- $\delta_1(H)$ = minimum vertex degree
- $\delta_{k-1}(H)$ = minimum codegree

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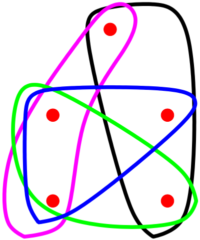

$\delta_1(H) = 2$ and $\delta_2(H) = 1$

Theorem (Daykin and Häggkvist 1981)

Suppose H k-uniform hypergraph, $|H| = n$ where $k|n$

$$\delta_1(H) \geq (1 - 1/k)\binom{n-1}{k-1} \implies \text{perfect matching}$$

- Condition on $\delta_1(H)$ believed to be far from best possible.

Theorem (Hán, Person and Schacht 2009)

$\forall\ \varepsilon > 0\ \exists\ n_0 \in \mathbb{N}$ *s.t if H 3-uniform, $n := |H| \geq n_0$ and*

$$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2} + \varepsilon n^2$$

$\implies$ *perfect matching*

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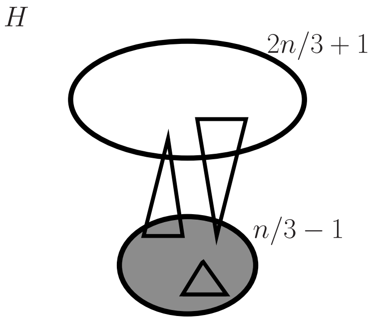
- Result best possible up to error term εn^2



$$\delta_1(H) = \binom{n-1}{2} - \binom{2n/3}{2}$$

no perfect matching

Theorem (Kühn, Osthus and T.)

$\exists\ n_0 \in \mathbb{N}$ s.t if H 3-uniform, $n := |H| \geq n_0$ and

$$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2}$$

then H contains a perfect matching.

- Independently, Khan proved this result.
- In fact, we prove a much stronger result...

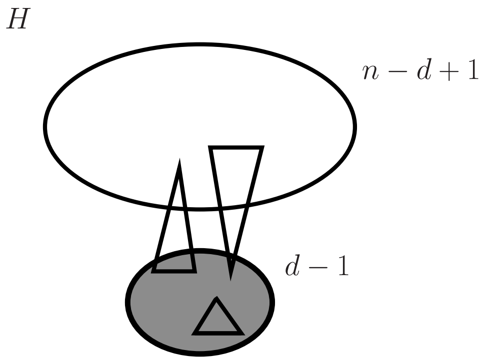
Theorem (Kühn, Osthus and T.)

$\exists\ n_0 \in \mathbb{N}$ s.t if H 3-uniform, $n := |H| \geq n_0$, $1 \leq d \leq n/3$ and

$$\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2}$$

then H contains a matching of size at least d.

- Bollobás, Daykin and Erdős (1976) proved result in case when $d < n/54$
- Result is tight



$$\delta_1(H) = \binom{n-1}{2} - \binom{n-d}{2}$$

no *d*-matching

More recent developments

- Khan (2011+) determined the exact minimum vertex degree which forces a perfect matching in a 4-uniform hypergraph.
- Alon, Frankl, Huang, Rödl, Ruciński, Sudakov (2012) gave asymptotically exact threshold for 5-uniform hypergraphs.
- No other *exact* vertex degree results are known. (Best known general bounds are due to Markström and Ruciński (2011).)

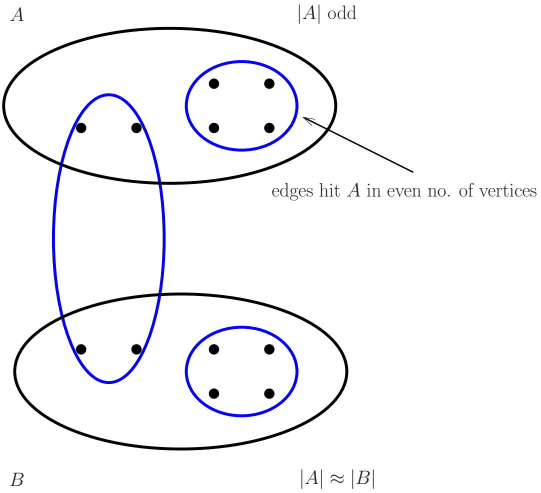
Theorem (Rödl, Ruciński and Szemerédi 2009)

H k-uniform hypergraph, $|H| = n$ sufficiently large, $k|n$

$$\delta_{k-1}(H) \geq n/2 \implies \textit{perfect matching}$$

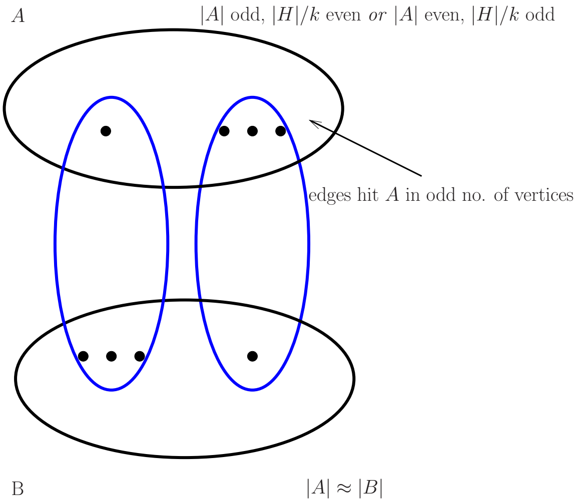
- In fact, they gave exact minimum codegree threshold that forces a perfect matching.

Type 1



$\delta_{k-1}(H) \approx |H|/2$ but no perfect matching

Type 2



$\delta_{k-1}(H) \approx |H|/2$ but no perfect matching

Theorem (Pikhurko 2008)

Suppose H k-uniform hypergraph on n vertices and $k/2 \leq \ell \leq k-1$.

$$\delta_\ell(H) \geq (1/2 + o(1))\binom{n-\ell}{k-\ell} \implies \textit{perfect matching}$$

- Previous examples shows result essentially best-possible.

Let $\delta(n, k, \ell)$ denote the max. value of $\delta_\ell(H)$ amongst all k-uniform hypergraphs H on n vertices of Type 1 or 2.

Theorem (T., Zhao)

Let n be sufficiently large. Suppose H k-uniform hypergraph on n vertices and $k/2 \leq \ell \leq k-1$.

$$\delta_\ell(H) > \delta(n, k, \ell) \implies \textit{perfect matching}$$

- Our result makes Pikhurko's exact.
- Our result implies the theorem of Rödl, Ruciński and Szemerédi.

Proof overview

We will only consider the case of 4-uniform hypergraphs and minimum 2-degree.

- H 4-uniform on n vertices and $\delta_2(H) > \delta(n, 4, 2)$

Absorbing sets

Let $0 < \varepsilon \ll \gamma \ll 1$.

- $S \subseteq V(H)$ an absorbing set if
  - $|S| = \gamma n$ and $H[S]$ contains a perfect matching
  - $H[S \cup Q]$ has a perfect matching for *any* set $Q \subseteq V(H)$ s.t. $|Q| \leq \varepsilon n$.

Theorem (Markström and Ruciński)

Suppose H 4-uniform on n vertices

$$\delta_2(H) \geq \left(\frac{7}{16} + o(1)\right)\binom{n}{2} \quad \Longrightarrow$$

H contains matching covering all but $\sqrt{n}$ vertices.

Our proof is therefore easy if we have an absorbing set:

- Find absorbing set S in H
- Find a matching M in $H - S$ covering almost all vertices
- Absorb uncovered vertices using S to obtain perfect matching

One can show that there is an absorbing set if:

(i) $\forall\ xy \in \binom{V(H)}{2}$, $\exists\ (1/2+o(1))\binom{n}{2}$ tuples $ab \in \binom{V(H)}{2}$ s.t. $|N_H(xy) \cap N_H(ab)| \geq o(1)n^2$ *or*

(ii) $\exists\ o(1)n^2$ pairs $xy \in \binom{V(H)}{2}$ of “large degree”, i.e. $d_H(xy) \geq (1/2+o(1))\binom{n}{2}$.

We can therefore assume (i) and (ii) fail.

We will show that this means H is close to one of the extremal hypergaphs (Type 1 or 2).

Open problems

- Characterise the minimum vertex degree that forces a perfect matching in a k-uniform hypergraph for $k \geq 5$.
- What about minimum ℓ-degree conditions for k-uniform H where $1 < \ell < k/2$?
  (Alon, Frankl, Huang, Rödl, Ruciński, Sudakov have some such results.)
- Establish k-partite analogues of the known results.