Tilings in graphs and directed graphs

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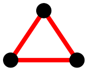
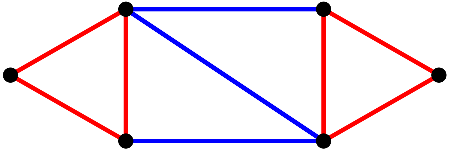
Includes joint work with Andrzej Czygrinow, Louis DeBiasio and Theo Molla and; József Balogh and Adam Zsolt Wagner.

Perfect tilings in graphs

- An *H-tiling* in G is a collection of vertex-disjoint copies of H in G.
- An H-tiling is *perfect* if it covers all vertices in G.

H





perfect H-packing

Remarks

- Perfect H-tilings sometimes called H-factors or perfect H-packings.
- If $H = K_2$ then perfect H-tiling $\iff$ perfect matching.
- Decision problem NP-complete (Hell and Kirkpatrick 1983).
- Sensible to look for simple sufficient conditions.

Perfect K_r-tilings

Theorem (Hajnal, Szemerédi 1970)

G graph, $|G| = n$ where $r|n$ and

$$\delta(G) \geq (1 - 1/r)\, n$$

$\Rightarrow$ G contains a perfect K_r-tiling.

- Corrádi and Hajnal (1964) proved triangle case.
- Easy to see minimum degree condition tight.

Perfect H-tilings for general H

Theorem (Alon and Yuster 1996)

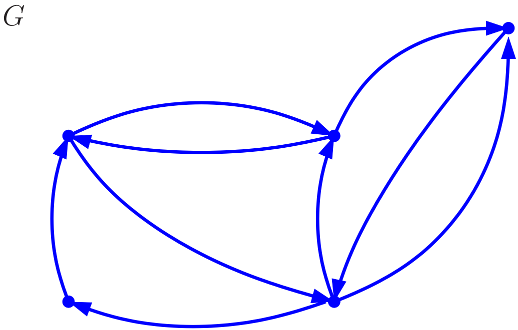
Let H be a graph with $\chi(H) = r$. Suppose G graph, $|G| = n$ where $|H||n$ and

$$\delta(G) \geq (1 - 1/r + o(1))n$$

$\Rightarrow$ G contains a perfect H-tiling.

- Result best-possible up to error term $o(1)n$ for many graphs H.
- Komlós, Sárközy and Szemerédi '01 replaced error term with a constant dependent on H.
- Kühn and Osthus '09 characterised, up to an additive constant, $\delta(G)$ that forces perfect H-tiling for *any* H.

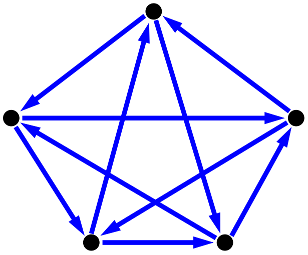
Our digraphs are allowed "double edges".



$\delta^+(G) = \delta^-(G) = 1$

The Hajnal-Szemerédi theorem for directed graphs

- Minimum semi-degree $\delta^0(G) := \min\{\delta^+(G), \delta^-(G)\}$
- Minimum total degree $\delta(G)$ = minimum number of edges incident to a vertex in G
- Tournament: orientation of a complete graph



- T_r = transitive tournament on r vertices
- C_3 = cyclic triangle

A minimum semi-degree result

Theorem (T. 2016)

G large digraph, $|G| = n$ where $r|n$. Let T be tournament on r vertices.

$$\delta^0(G) \geq (1 - 1/r)n$$

$\Rightarrow$ *G contains a perfect T-tiling.*

- Minimum semi-degree condition best-possible.
- Earlier, Czygrinow, Kierstead and Molla gave approximate result when $T = C_3$.
- Result implies the Hajnal-Szemerédi theorem for large graphs.

A minimum semi-degree result

Theorem (T. 2016)

G large digraph, $|G| = n$ where $r|n$. Let T be tournament on r vertices.

$$\delta^0(G) \geq (1 - 1/r)n$$

$\Rightarrow$ *G contains a perfect T-tiling.*

- Natural to ask if we can replace condition here with $\delta(G) \geq 2(1 - 1/r)n - 1$.
- However, a result of Wang shows we cannot do this for $T = C_3$.

Minimum total degree results

Theorem (Czygrinow, DeBiasio, Kierstead and Molla 2015)

G digraph, $|G| = n$ where $r|n$.

$$\delta(G) \geq 2(1 - 1/r)n - 1$$

$\Rightarrow$ G contains a perfect T_r-tiling.

- Minimum degree condition best-possible.
- Implies the Hajnal-Szemerédi theorem.

Minimum total degree results

Theorem (Czygrinow, DeBiasio, Molla and T. 2016+)

Let $r \geq 4$, and T be tournament on r vertices. If G large digraph, $|G| = n$ where $r|n$ s.t.

$$\delta(G) \geq 2(1 - 1/r)n - 1$$

$\Rightarrow$ *G contains a perfect T-tiling.*

- Degree condition best-possible.
- Actually, we prove a signficantly stronger result.

Minimum total degree results

Theorem (Czygrinow, DeBiasio, Molla and T. 2016+)

Let $r \geq 4$. If G large digraph, $|G| = n$ where $r|n$ s.t.

$$\delta(G) \geq 2(1 - 1/r)n - 1$$

$\Rightarrow$ G contains n/r vertex-disjoint subdigraphs each of which contains every tournament on r vertices.

- We prove this result by translating the problem into the multigraph setting.

Perfect tilings in random graphs

The Erdős–Renyi graph $G_{n,p}$ has:

- Vertex set $[n] := \{1, \ldots, n\}$;
- Each edge is present with probability p, independently of all other choices.

Given a graph H define

$$d(H) := \frac{e(H)}{|H|-1} \quad \text{and} \quad d^*(H) := \max\{d(H') : H' \subseteq H, |H'| \geq 2\}.$$

If $d(H') < d(H)$ for all $H' \subset H$ we say H is strictly balanced.

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If $d(H') < d(H)$ for all $H' \subset H$ we say H is strictly balanced.

Theorem (Johansson, Kahn and Vu 2008)

H strictly balanced, $e(H) = m$. Let n be divisible by $|H|$.

- *If $p \gg n^{-1/d(H)}(\log n)^{1/m}$ then a.a.s. $G_{n,p}$ contains a perfect H-tiling.*
- *If $p \ll n^{-1/d(H)}(\log n)^{1/m}$ then a.a.s. $G_{n,p}$ does not contain a perfect H-tiling.*

Gerke and McDowell (2015) resolved case of 'nonvertex-balanced' graphs H. (Threshold $= n^{-1/d^*(H)}$.)

Randomly peturbed dense graphs

Bohman, Frieze and Martin (2003) introduced the concept of a randomly perturbed dense graph.

Framework:

- Start with any n-vertex G with $\delta(G) \geq \alpha n$ ($\alpha > 0$ fixed).
- Consider $G \cup G_{n,p}$
- For which values of p does $G \cup G_{n,p}$ a.a.s. contain some substructure?

Randomly perturbed dense graphs

Threshold known for:

- Hamiltonicity (Bohman, Frieze and Martin 2003)
- Fixed spanning trees of bounded degree (Krivelevich, Kwan and Sudakov 2017)
- Fixed bounded degree spanning subgraph (Böttcher, Montgomery, Parczyk and Person 2017+)

Also generalised to digraph and hypergraph setting (Krivelevich, Kwan and Sudakov; McDowell and Mycroft)

Theorem (Balogh, T., Wagner 2017+)

Let H and $\alpha > 0$ be fixed. There is a $C = C(\alpha, H) > 0$ s.t. if $p \geq Cn^{-1/d^*(H)}$ and G is an n-vertex graph with

$$\delta(G) \geq \alpha n$$

$\implies$ *a.a.s. $G \cup G_{n,p}$ contains a perfect H-tiling.*

- Note H doesn't have to be strictly balanced.
- Save a log-term here compared to Johansson–Kahn–Vu.
- Probability threshold best possible.

Open problems

Digraphs:

- Perfect H-tilings in digraphs where H *is not a tournament*
- Perfect tilings in oriented graphs (very little known!!)

Random perturbed dense graphs:

- Lower our linear minimum degree term?
- What about randomly perturbed 'very dense' graphs?