A random version of Sperner's theorem

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Joint work with
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Birmingham, September 18-20 2015

- Plenary speakers: Noga Alon, Keith Ball, Béla Bollobás, Timothy Gowers, Stefanie Petermichl, and Aner Shalev
- Invited Combinatorics speakers: József Balogh, Mihyun Kang, Michael Krivelevich, Marc Noy, Wojciech Samotij, Mathias Schacht, Benny Sudakov
- Chance for postdocs to give talks, student poster session
- Registration: Free for students, £20 otherwise.

http://web.mat.bham.ac.uk/emslmsweekend/

Random versions of classical theorems

Recently, there has been a focus on developing *random* analogues of classical theorems in Combinatorics:

- Ramsey's theorem: Frankl, Rödl, Łuczak, Ruciński, Voigt, Conlon, Gowers, Friedgut, Tetali...
- Turán's theorem: Haxell, Kohayakawa, Łuczak, Schacht, Conlon, Gowers, Balogh, Morris, Samotij...
- Erdős–Ko–Rado theorem: Balogh, Bohman, Mubayi, Hamm, Kahn,...
- Szemerédi's theorem: Kohayakawa, Łuczak, Rödl, Schacht, Conlon, Gowers, Balogh, Morris, Samotij ...

See survey 'Combinatorial theorems relative to a random set' (Conlon) for more details.

Antichains and Sperner's theorem

- $[n] := \{1, \ldots, n\}$
- $\mathcal{P}(n)$ denotes power set of $[n]$
- $\mathcal{A} \subseteq \mathcal{P}(n)$ antichain if $\nexists\ A, B \in \mathcal{A}$ s.t. $A \subset B$

Theorem (Sperner, 1928)

The largest antichain in $\mathcal{P}(n)$ has size $\binom{n}{\lfloor n/2 \rfloor}$.

The Random model $\mathcal{P}(n, p)$

- $\mathcal{P}(n, p)$ is obtained from $\mathcal{P}(n)$ by selecting each element of $\mathcal{P}(n)$ with probability p
- Model first considered by Rényi (1961) who determined the probability threshold for the property that $\mathcal{P}(n, p)$ is not an antichain itself

Question (Kohayakawa and Kreuter)

For what values of p does the following hold? With high probability, the largest antichain in $\mathcal{P}(n, p)$ has size

$$(1 + o(1))p\binom{n}{n/2}.$$

Previous results

Proposition (Osthus 2000)

Suppose $p = c/n$ where $c > 0$ is fixed. Whp largest antichain in $\mathcal{P}(n,p)$ has size at least

$$(1+o(1))(1+e^{-c/2})p\binom{n}{n/2}.$$

Theorem (Osthus 2000)

If $pn/\log n \to \infty$, then whp the largest antichain in $\mathcal{P}(n,p)$ has size

$$(1+o(1))p\binom{n}{n/2}.$$

A random version of Sperner's theorem

Theorem (Balogh, Mycroft, T. 2014)

$\forall\ \varepsilon > 0$, $\exists\ C$ s.t. if $p > C/n$ then whp largest antichain in $\mathcal{P}(n,p)$ has size at most

$$(1+\varepsilon)p\binom{n}{n/2}.$$

- Completely solves Kohayakawa–Kreuter question
- Independently proven by Collares Neto and Morris

Theorem (Balogh, Mycroft, T. 2014)

$\forall\ \varepsilon > 0$, $\exists\ C$ s.t. if $p > C/n$ then whp largest antichain in $\mathcal{P}(n,p)$ has size at most

$$(1+\varepsilon)p\binom{n}{n/2}.$$

Naïve strategy:

- If $\mathcal{A}$ antichain then whp intersection of $\mathcal{A}$ in $\mathcal{P}(n,p)$ is $(1 \pm \varepsilon)p|\mathcal{A}|$.
- Sum up these events
- Problem is there are too many events!! (Kleitman: $2^{(1+o(1))\binom{n}{n/2}}$ antichains)

Proof strategy

Lemma

There is a collection $\mathcal{F}$ where each $F \in \mathcal{F}$ is a subset of $\mathcal{P}(n)$ and

(i) $|\mathcal{F}| = o(2^{\binom{n}{n/2}})$;

(ii) $|F| \leq (1+\varepsilon/2)\binom{n}{n/2}$ *for all* $F \in \mathcal{F}$;

(iii) *Every antichain lies in some element of* $\mathcal{F}$.

- Example of a Container result
- (i)-(ii) ensures that whp $\mathcal{P}(n,p)$ contains at most $(1+\varepsilon)p\binom{n}{n/2}$ elements from F for all $F \in \mathcal{F}$;
- (iii) implies whp every antichain in $\mathcal{P}(n,p)$ has size at most $(1+\varepsilon)p\binom{n}{n/2}$

Proof of Container lemma

Lemma

There is a collection $\mathcal{F}$ where each $F \in \mathcal{F}$ is a subset of $\mathcal{P}(n)$ and

(i) $|\mathcal{F}| = o(2^{\binom{n}{n/2}})$;

(ii) $|F| \leq (1+\varepsilon/2)\binom{n}{n/2}$ *for all* $F \in \mathcal{F}$;

(iii) *Every antichain lies in some element of* $\mathcal{F}$.

Define auxiliary graph G where:

- $V(G) = \mathcal{P}(n)$;
- A and B are adjacent if and only if $A \subset B$ or $B \subset A$.

So the independent sets in G are precisely the antichains in $\mathcal{P}(n)$.

Algorithm for Container lemma

Fix total ordering $v_1, \ldots, v_{2^n}$ of vertices in G.
Fix independent set I in G

Algorithm:

Let $G_0 = G$; $S = \emptyset$.

Step i:

Let $u \in V(G_{i-1})$ s.t. $d_{G_{i-1}}(u) = \Delta(G_{i-1})$

- If $u \notin I$ set $G_i := G_{i-1} \setminus \{u\}$.
- If $u \in I$ and $d_{G_{i-1}}(u) \geq \varepsilon n$ **add** u to S and let $G_i := G_{i-1} \setminus (\{u\} \cup N_G(u))$.
- If $u \in I$ and $d_{G_{i-1}}(u) < \varepsilon n$ **add** u to S, set $G_i := G_{i-1} \setminus \{u\}$ and **stop**.

At end define container $F := S \cup V(G_i)$.

Open problem

Question

For what values of p does the following hold? With high probability, the largest antichain in $\mathcal{P}(n, p)$ consists precisely of the elements of the middle layer.

- Hamm and Kahn (2014+) have answered question in the affirmative for $p > 1 - \varepsilon$ for some fixed $\varepsilon > 0$.