A random version of Sperner's theorem

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Abstract

Let $\mathcal{P}(n)$ denote the power set of $[n]$, ordered by inclusion, and let $\mathcal{P}(n,p)$ be obtained from $\mathcal{P}(n)$ by selecting elements from $\mathcal{P}(n)$ independently at random with probability p. A classical result of Sperner [12] asserts that every antichain in $\mathcal{P}(n)$ has size at most that of the middle layer, $\binom{n}{\lfloor n/2 \rfloor}$. In this note we prove an analogous result for $\mathcal{P}(n,p)$: If $pn \to \infty$ then, with high probability, the size of the largest antichain in $\mathcal{P}(n,p)$ is at most $(1+o(1))p\binom{n}{\lfloor n/2 \rfloor}$. This solves a conjecture of Osthus [9] who proved the result in the case when $pn/\log n \to \infty$. Our condition on p is best-possible. In fact, we prove a more general result giving an upper bound on the size of the largest antichain for a wider range of values of p.

We write $[n]$ for the set of natural numbers up to n, and $\mathcal{P}(n)$ for the power set of $[n]$. Also, for any $0 \leq k \leq n$ we write $\binom{[n]}{k}$ for the subset of $\mathcal{P}(n)$ consisting of all sets of size k. A subset $\mathcal{A} \subseteq \mathcal{P}(n)$ is an *antichain* if for any $A, B \in \mathcal{A}$ with $A \subseteq B$ we have $A = B$. So $\binom{[n]}{k}$ is an antichain for any $0 \leq k \leq n$; Sperner's theorem [12] states that in fact no antichain in $\mathcal{P}(n)$ has size larger than $\binom{n}{\lfloor n/2 \rfloor}$. Our main theorem is a random version of Sperner's theorem. For this, let $\mathcal{P}(n,p)$ be the set obtained from $\mathcal{P}(n)$ by selecting elements randomly with probability p and independently of all other choices. Write $m := \binom{n}{\lfloor n/2 \rfloor}$. Roughly speaking, our main result asserts that if $p > C/n$ for some constant C, then with high probability, the largest antichain in $\mathcal{P}(n,p)$ is approximately the same size as the 'middle layer' in $\mathcal{P}(n,p)$.

Theorem 1. *For any $\varepsilon > 0$ there exists a constant C such that if $p > C/n$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at most $(1+\varepsilon)pm$.*

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(Here, by 'with high probability' we mean with probability tending to 1 as n tends to infinity.)

The model $\mathcal{P}(n,p)$ was first investigated by Rényi [10] who determined the probability threshold for the property that $\mathcal{P}(n,p)$ is not itself an antichain, thereby answering a question of Erdős. The size of the largest antichain in $\mathcal{P}(n,p)$ for p above this threshold was first studied by Kohayakawa and Kreuter [6]. In [6] they raised the question of which values of p does the conclusion of Theorem 1 hold. Osthus [9] proved Theorem 1 in the case when $pn/\log n \to \infty$ and conjectured that this can be replaced by $pn \to \infty$. (So Theorem 1 resolves this conjecture.) Moreover, Osthus showed that, for a fixed $c > 0$, if $p = c/n$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at least $(1+o(1))(1+e^{-c/2})p\binom{n}{\lfloor n/2 \rfloor}$. So the bound on p in Theorem 1 is best-possible up to the constant C. There have also been a number of results concerning the length of (the longest) chains in $\mathcal{P}(n,p)$ and related models of random posets (see for example, [2, 7, 8]).

Instead of proving Theorem 1 directly we prove the following more general result.

Theorem 2. *Let $n \in \mathbb{N}$ and $m := \binom{n}{\lfloor n/2 \rfloor}$. For any $\varepsilon > 0$ and $t \in \mathbb{N}$, there exists a constant C such that if $p > C/n^t$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at most $(1+\varepsilon)pmt$.*

Osthus [9] proved this result in the case when $p(n/t)^t/\log n \to \infty$. (In fact, Osthus's result allows for t to be an integer function, see [9] for the precise statement.) Moreover, Osthus showed that, for $1/n^t \ll p \ll 1/n^{t-1}$, with high probability, $\mathcal{P}(n,p)$ has an antichain of size at least $(1+o(1))pmt$ (so Theorem 2 is 'tight' in this window of p).

The method of proof of Theorem 2 also allows us to estimate the number of antichains in $\mathcal{P}(n)$ of certain fixed sizes.

Proposition 3. *Fix any $t \in \mathbb{N}$, and suppose that $m/n^t \ll s \ll m/n^{t-1}$. Then the number of antichains of size s in $\mathcal{P}(n)$ is $\binom{(t+o(1))m}{s}$.*

To prove Theorem 2, let G be the graph with vertex set $\mathcal{P}(n)$ in which distinct sets A and B are adjacent if $A \subseteq B$ or $B \subseteq A$. Then an antichain in $\mathcal{P}(n)$ is precisely an independent set in G. We follow the 'hypergraph container' approach (see, for example, [1, 11]): indeed, we show that all independent sets in G are contained within a fairly small number of low-density sets in G. Crucially, for this method to work, we have to construct our 'containers' in two phases (see Lemma 6). For this we use a result of Kleitman [5] on the minimum number of edges induced by a subset of G with a given fixed size. Define the *centrality order* on the vertices of $\mathcal{P}(n)$ as follows: we begin with the elements of $\binom{[n]}{\lfloor n/2 \rfloor}$, ordered arbitrarily, then the elements of $\binom{[n]}{\lfloor n/2 \rfloor+1}$, then the elements of $\binom{[n]}{\lfloor n/2 \rfloor-1}$, then the elements of $\binom{[n]}{\lfloor n/2 \rfloor+2}$, and so forth until all vertices of $\mathcal{P}(n)$ have been ordered. For any $r \in \mathbb{N}$ let I_r denote the initial segment of this order of length r; Kleitman [5] proved that I_r minimises the number of induced edges over all sets of size r (see also [4], which characterises all the sets U of size r for which $e(G[U])$ is minimised).

Theorem 4 (Kleitman [5])**.** *For any $r \leq 2^n$ and any $U \subseteq V(G)$ of size r we have $e(G[U]) \geq e(G[I_r])$.*

We apply this theorem in the form of the following corollary.

Corollary 5. *Let $U \subseteq V(G)$, and suppose that $0 < \varepsilon \leq 1/2$ and $t \in \mathbb{N}$. If $|U| \geq (t+\varepsilon)m$, then $e(G[U]) > \varepsilon n^t |U|/(2t)^{t+1}$.*

Proof. Let $r := |U|$. We have $r \geq (t+\varepsilon)m$, so in particular $r - mt \geq r(1 - t/(t+\varepsilon)) \geq 2\varepsilon r/(1+2t)$ since $\varepsilon \leq 1/2$. Observe that I_r contains all of the at most mt elements of the t 'middle layers', $\binom{[n]}{\lfloor n/2 \rfloor}$, $\binom{[n]}{\lfloor n/2 \rfloor + 1}$, and so forth. Further, I_r contains at least $r - mt$ elements from outside these layers, each of which has at least $\binom{\lceil n/2 \rceil}{t} \geq (n/2t)^t$ neighbours in the t middle layers. So by Theorem 4 we have

$$e(G[U]) \geq e(G[I_r]) \geq \frac{2\varepsilon r}{1+2t} \cdot \left(\frac{n}{2t}\right)^t \geq \frac{\varepsilon n^t r}{(2t)^{t+1}}. \qquad \square$$

Let $s \in \mathbb{N}$, $t > 0$ and let S be a set of size $|S| = s$. Define $\binom{S}{\leq t}$ to be the set of all subsets of S of size at most t and $\binom{s}{\leq t} := \left|\binom{S}{\leq t}\right|$.

Lemma 6. *Suppose that $t \in \mathbb{N}$, $0 < \varepsilon \leq 1/(2t)^{t+1}$ and n is sufficiently large. Then there exist functions $f : \binom{V(G)}{\leq n^{-(t+0.9)}2^n} \to \binom{V(G)}{\leq (t+1+\varepsilon)m}$ and $g : \binom{V(G)}{\leq (t+2)m/(\varepsilon^2 n^t)} \to \binom{V(G)}{\leq (t+\varepsilon)m}$ such that, for any independent set I in G, there are disjoint subsets $S_1, S_2 \subseteq I$ with $S_1 \in \binom{V(G)}{\leq n^{-(t+0.9)}2^n}$, $S_2 \in \binom{V(G)}{\leq (t+2)m/(\varepsilon^2 n^t)}$ such that $S_1 \cup S_2$ and $g(S_1 \cup S_2)$ are disjoint, $S_2 \subseteq f(S_1)$, and $I \subseteq S_1 \cup S_2 \cup g(S_1 \cup S_2)$.*

Roughly speaking, Lemma 6 ensures that every independent set I in G lies in some (not too big) sparse 'container' set $S_1 \cup S_2 \cup g(S_1 \cup S_2)$, and in total we do not have 'too many' containers. Indeed, since S_1 and S_2 are small sets, there are not too many possibilities for the set $S_1 \cup S_2$, which in turn means there are not too many containers $S_1 \cup S_2 \cup g(S_1 \cup S_2)$ to consider. This property is crucial to the proof of Theorem 2, as it enables us to take a union bound to show that it is unlikely that the number of vertices randomly selected from any container is significantly higher than expected.

Proof of Lemma 6. Fix an arbitrary total order $v_1, \ldots, v_n$ on the vertices of $V(G)$. Given any independent set I in G, define $G_0 := G$, and take S_1 and S_2 to be initially empty. We add vertices to S_1 and S_2 through the following iterative process, beginning at Step 1 in Phase 1.

Phase 1: At Step i, let u be the maximum degree vertex of G_{i-1} (with ties broken by our fixed total order). If $u \notin I$ then define $G_i := G_{i-1} \setminus \{u\}$, and proceed to Step $i+1$ (still in Phase 1). Alternatively, if $u \in I$ and $\deg_{G_{i-1}}(u) \geq n^{t+0.9}$ then add u to S_1, define $G_i := G_{i-1} \setminus (\{u\} \cup N_G(u))$, and proceed to Step $i+1$ (still in Phase 1). Finally, if $u \in I$ and $\deg_{G_{i-1}}(u) < n^{t+0.9}$, then add u to S_1, define $G_i := G_{i-1} \setminus \{u\}$ and $f(S_1) := V(G_i)$, and proceed to Step $i+1$ of Phase 2.

Phase 2: At Step i, let u be the maximum degree vertex of G_{i-1}. If $u \notin I$ then define $G_i := G_{i-1} \setminus \{u\}$, and proceed to Step $i+1$ (still in Phase 2). Alternatively, if $u \in I$ and $\deg_{G_{i-1}}(u) \geq \varepsilon^2 n^t$ then add u to S_2, define $G_i := G_{i-1} \setminus (\{u\} \cup N_G(u))$, and proceed to Step

$i+1$ (still in Phase 2). Finally, if $u \in I$ and $\deg_{G_{i-1}}(u) < \varepsilon^2 n^t$, then add u to S_2, define $G_i := G_{i-1} \setminus \{u\}$ and $g(S_1 \cup S_2) := V(G_i)$, and terminate.

Observe first that for any independent set I in G the process defined ensures that S_1 and S_2 are disjoint subsets of I, that $S_1 \cup S_2$ is disjoint from $g(S_1 \cup S_2)$, that $S_2 \subseteq f(S_1)$ and that $I \subseteq S_1 \cup S_2 \cup g(S_1 \cup S_2)$.

Next, note that for any independent set I, if a vertex u is added to S_1 at step i, u and at least $n^{t+0.9}$ neighbours of u are deleted from G_{i-1} in forming G_i, with a single exception (when u is the final vertex added to S_1). So we must have $|S_1| \leq 1 + |V(G)|/(n^{t+0.9}+1) \leq n^{-(t+0.9)}2^n$. Furthermore, at the end of Phase 1 we know that every vertex v of G_i has $\deg_{G_i}(v) \leq n^{t+0.9}$, and so Corollary 5 implies that $f(S_1)$, the set of all vertices not deleted up to this point, must have size $|f(S_1)| < (t+1+\varepsilon)m$. Then, in Phase 2, if a vertex u is added to S_2 at step i, at least $\varepsilon^2 n^t$ neighbours of u are deleted from G_{i-1} in forming G_i, again with the single exception of the final vertex added to S_2. So we must have $|S_2| \leq 1 + |f(S_1)|/(\varepsilon^2 n^t)$ and thus

$$|S_1 \cup S_2| \leq 1 + (t+1+\varepsilon)m/(\varepsilon^2 n^t) + n^{-(t+0.9)}2^n \leq (t+2)m/(\varepsilon^2 n^t).$$

Moreover, at the end of Phase 2 every vertex v of the final G_i has $\deg_{G_i}(v) \leq \varepsilon^2 n^t$ and so $e(G_i) \leq \varepsilon^2 n^t |G_i| \leq \varepsilon n^t |G_i|/(2t)^{t+1}$. Thus, Corollary 5 implies that $|g(S_1 \cup S_2)| \leq (t+\varepsilon)m$.

So it is sufficient to check that the functions f and g are well-defined. That is, we must check that if the process described above yields the same set S_1 when applied to independent sets I and I', then it should also yield the same set $f(S_1)$, and if additionally the same set S_2 is returned then the sets $g(S_1 \cup S_2)$ should be identical. However, this is a consequence of the fact that we always chose u to be the vertex of I of maximum degree in G_{i-1}. Moreover, if our algorithm produces sets S_1, S_2 for an independent set I and sets S_1', S_2' for an independent set I' such that $S_1 \cup S_2 = S_1' \cup S_2'$ then $S_1 = S_1'$ (and $S_2 = S_2'$). Thus, indeed f and g are well-defined. □

The reason for using a two-phase algorithm in the proof of Lemma 6 is that the structure of the hypercube graph is locally highly asymmetric; even worse, the size of the targeted independent set I is very small compared to the number of vertices in the graph. Roughly speaking, the main objective of Phase 1 (where in each step many vertices are removed) is to decrease the number of potential vertices of I sufficiently for the standard 'hypergraph container' approach of Phase 2 to be successful.

Proof of Theorem 2. Fix $\varepsilon > 0$ and $t \in \mathbb{N}$; we may assume that $\varepsilon < 1/(2t)^{t+1}$. Define $C := 10^{10}\varepsilon^{-5}$ and $\varepsilon_1 := \varepsilon/4$. Let G_p be the graph formed from G by selecting vertices independently at random with probability $p > C/n^t$. Then we must show that, with high probability, G_p has no independent set of size greater than $(1+\varepsilon)pmt$. Apply Lemma 6 with ε_1 playing the role of ε. Suppose for a contradiction that G_p does contain some independent set I with $|I| > (1+\varepsilon)pmt$. Then all vertices of the sets S_1 and S_2 given by Lemma 6 for this I must have been selected for G_p, along with at least $|I| - |S_1 \cup S_2| \geq (1+\varepsilon)pmt - (t+2)m/(\varepsilon_1^2 n^t) \geq (1+\varepsilon/2)pmt$ vertices of $g(S_1 \cup S_2)$ (the second inequality follows from $C = 10^{10}\varepsilon^{-5}$).

However, the number of possibilities for S_1 is $\binom{2^n}{\leq n^{-(t+0.9)}2^n}$, and for each possibility the probability that $S_1 \subseteq V(G_p)$ is $p^{|S_1|}$. For any fixed S_1 we have $|f(S_1)| \leq (t+2)m$ and

$S_2 \subseteq f(S_1)$, so the number of possibilities for S_2 is at most $\binom{(t+2)m}{\le (t+2)m/(\varepsilon_1^2 n^t)}$, and for each possibility the probability that $S_2 \subseteq V(G_p)$ is $p^{|S_2|}$. Finally, for any fixed S_1 and S_2 we have $g(S_1 \cup S_2) \le (t+\varepsilon_1)m \le (1+\varepsilon/4)mt$, so the expected number of vertices of $g(S_1 \cup S_2)$ selected for G_p is at most $(1+\varepsilon/4)pmt$. By a standard Chernoff bound the probability that at least $(1+\varepsilon/2)pmt$ vertices of $g(S_1 \cup S_2)$ are selected for G_p is therefore at most $e^{-\varepsilon^2 pmt/100}$. Taking a union bound, we conclude that the probability that G_p contains an independent set I of size greater than $(1+\varepsilon)pmt$ is at most

$$\begin{aligned}\Pi &:= \sum_{0 \le a \le n^{-(t+0.9)}2^n} \sum_{0 \le b \le (t+2)m/(\varepsilon_1^2 n^t)} \binom{2^n}{a} \cdot p^a \cdot \binom{(t+2)m}{b} \cdot p^b \cdot e^{-\varepsilon^2 pmt/100} \\ &\le (n^{-(t+0.9)}2^n + 1)((t+2)m/(\varepsilon_1^2 n^t)+1) \binom{2^n}{n^{-(t+0.9)}2^n} \cdot p^{n^{-(t+0.9)}2^n} \binom{(t+2)m}{(t+2)m/(\varepsilon_1^2 n^t)} \\ &\quad \cdot p^{(t+2)m/(\varepsilon_1^2 n^t)} \cdot e^{-\varepsilon^2 pmt/100}.\end{aligned}$$

Note that for large n, with plenty of room to spare we have

$$(n^{-(t+0.9)}2^n + 1)((t+2)m/(\varepsilon_1^2 n^t)+1) \le e^{\varepsilon^2 pmt/400}$$

and

$$\binom{2^n}{n^{-(t+0.9)}2^n} \cdot p^{n^{-(t+0.9)}2^n} \le e^{\varepsilon^2 pmt/400}.$$

Further, since $C = 10^{10}\varepsilon^{-5}$, for large n we have that

$$\binom{(t+2)m}{(t+2)m/(\varepsilon_1^2 n^t)} \cdot p^{(t+2)m/(\varepsilon_1^2 n^t)} \le e^{\varepsilon^2 pmt/400}.$$

Thus, the upper bound Π on the probability is $o(1)$. □

We conclude with a sketch of the proof of Proposition 3, on the number of antichains of given fixed sizes in $\mathcal{P}(n)$.

Proof sketch of Proposition 3. The lower bound can be obtained by greedily choosing vertices from within the t middle layers of $\mathcal{P}(n)$ to form an antichain of size s, and counting the number of ways to make these choices. For the upper bound, fix any $\varepsilon > 0$ and apply Lemma 6 with this ε and t. Then any independent set in G of size s is uniquely determined by the choice of

1. a set S_1 of size $s_1 \le \ell_1 := 2^n/n^{t+0.9}$, for which there are at most $\binom{2^n}{\le \ell_1}$ choices,
2. a set $S_2 \subseteq f(S_1)$ of size $s_2 \le \ell_2 := (t+2)m/(\varepsilon^2 n^t)$, for which there are at most $\binom{(t+1+\varepsilon)m}{\le \ell_2}$ choices, and
3. a set $S \subseteq g(S_1 \cup S_2)$ of size $s - s_1 - s_2$, for which there are at most $\binom{(t+\varepsilon)m}{s-s_1-s_2}$ choices.

Summing over all these choices by a similar calculation as in the proof of Theorem 2, we find that (for large n) there are at most $\binom{(t+2\varepsilon)m}{s}$ independent sets of size s in G. $\square$

When we completed the project, we were informed that Collares Neto and Morris [3] independently proved Theorem 1. Their method is however different. We used the proof technique of [1], and they followed the method of [11]. In particular, when we constructed containers, we aimed at having few vertices, whilst they aimed at having only few edges.

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