The complexity of perfect matchings and packings in dense graphs

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Joint work with
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In this talk we are interested in perfect matchings and packings in k-graphs H:

- perfect matchings = vertex-disjoint edges covering all of $V(H)$
- perfect F-packings = vertex-disjoint copies of F covering all of $V(H)$
- Edmonds' Algorithm: can find a perfect matching in a **graph** (if it exists) in polynomial time
- If $k \geq 3$ decision problem is NP-complete (Karp; Garey and Johnson)
- Graph perfect packings: decision problem is NP-complete, unless the packing corresponds to a matching (Hell, Kirkpatrick)

Overview

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Perfect matchings in k-graphs

Let H be a k-graph and $S \subseteq V(H)$.

- $d_H(S)$ = # edges containing S;
- $\delta_\ell(H) = \min\{d_H(S) : |S| = \ell\}$ (for fixed $1 \leq \ell \leq k-1$);
- $\delta_1(H)$ = minimum vertex degree;
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Conjecture (Hàn, Person Schacht; Kühn, Osthus)

Given an n-vertex k-graph H and fixed $1 \leq \ell \leq k-1$. If
$\delta_\ell(H) \geq \max\left\{\left(\frac{1}{2}+o(1)\right)\binom{n-\ell}{k-\ell}, \left(1-(1-\frac{1}{k})^{k-\ell}+o(1)\right)\binom{n-\ell}{k-\ell}\right\}$
$\implies$ *perfect matching in H.*

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Known for:

- $\ell = k-1$ (Rödl, Ruciński, Szemerédi)
- $\ell \ge k/2$ (Pikhurko; T. and Zhao)
- $\ell \ge 0.42k$ (Han)
- some small values of k, ℓ.

The decision problem

- Let $\mathbf{PM}(k, \ell, \delta)$ denote the decision problem of whether a k-graph H with $\delta_\ell(H) \geq \delta\binom{|H|-\ell}{k-\ell}$ contains a perfect matching.

Results:

- $\mathbf{PM}(k, k-1, 1/k)$ is in P
(Karpiński, Ruciński and Szymańska; Keevash, Knox and Mycroft; Han)
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Intuition: beyond 'space barrier' can decide in polynomial time

Conjecture (Keevash, Knox, Mycroft)

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Theorem (Han and T.)

Conjecture true for

$$(k-1)/2 \leq \ell \leq (1 + \log(2/3))k \approx 0.5945k$$

- Proof is one page consequence of a general black-box for matching and packing problems.
- If one solves the 'almost' perfect matching problem then our result immediately extends to all $\ell \leq (1 + \log(2/3))k$.

Perfect packings in graphs

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Theorem (Han and T.)

"Above space barrier we can always decide in polynomial time whether a graph contains a perfect F-packing."

- This answers a question of Yuster in the negative.