A NOTE ON SOME EMBEDDING PROBLEMS FOR ORIENTED GRAPHS

ANDREW TREGLOWN

ABSTRACT. We conjecture that every oriented graph G on n vertices with $\delta^+(G), \delta^-(G) \geq 5n/12$ contains the square of a Hamilton cycle. We also give a conjectural bound on the minimum semidegree which ensures a perfect packing of transitive triangles in an oriented graph. A link between Ramsey numbers and perfect packings of transitive tournaments is also considered.

1.1. Powers of Hamilton cycles.

One of the most studied problems in graph theory concerns finding sufficient conditions that ensure a graph contains a Hamilton cycle. Dirac [4] showed that any graph G on $n \geq 3$ vertices has a Hamilton cycle provided that it has minimum degree $\delta(G)$ at least $n/2$. For a digraph G it is natural to consider its *minimum semidegree* $\delta^0(G)$, which is the minimum of its minimum outdegree $\delta^+(G)$ and its minimum indegree $\delta^-(G)$. (The digraphs we consider do not have loops and we allow at most one edge in each direction between any pair of vertices.) Ghouila-Houri [6] proved that every digraph G on $n \geq 2$ vertices with $\delta^0(G) \geq n/2$ is Hamiltonian.

An important subclass of digraphs is the class of oriented graphs: these are the digraphs which do not contain any 2-cycles. Keevash, Kühn and Osthus [9] showed that any sufficiently large oriented graph G on n vertices with $\delta^0(G) \geq (3n-4)/8$ is Hamiltonian, thereby proving a conjecture of Häggkvist [7]. For a detailed account of other such results concerning Hamilton cycles in directed and oriented graphs see [13].

A generalisation of the notion of a Hamilton cycle is that of the rth power of a Hamilton cycle. Indeed, the rth power of a Hamilton cycle C is obtained from C by adding an edge between every pair of vertices of distance at most r on C. Seymour [18] conjectured the following strengthening of Dirac's theorem.

Conjecture 1 (Seymour [18]). *Let G be a graph on n vertices. If $\delta(G) \geq \frac{r}{r+1}n$ then G contains the rth power of a Hamilton cycle.*

Pósa (see [5]) had earlier proposed the conjecture in the case of the square of a Hamilton cycle (that is, when $r = 2$). Komlós, Sárközy and Szemerédi [11] proved Conjecture 1 for sufficiently large graphs.

The notion of the rth power of a Hamilton cycle also makes sense in the digraph setting: In this case the rth power of a Hamilton cycle C is the digraph obtained from C by adding a directed edge from x to y if there is a path of length at most r from x to y on C. Bollobás and Häggkvist [1] proved that given any $\varepsilon > 0$ and any $r \in \mathbb{N}$, all sufficiently large tournaments T on n vertices with $\delta^0(T) \geq (1/4 + \varepsilon)n$ contain the rth power of a Hamilton cycle.

One would expect that the minimum semidegree threshold that ensures a digraph contains the rth power of a Hamilton cycle is the 'same' as the condition in Conjecture 1. But it is far less clear at first sight what to expect in the oriented case. We propose the following oriented graph analogue of Pósa's conjecture.

Conjecture 2. *Suppose G is an oriented graph on n vertices such that $\delta^0(G) \geq 5n/12$. Then G contains the square of a Hamilton cycle.*

The following proposition shows that, if true, Conjecture 2 is 'best possible'.

Proposition 3. *Let $n \in \mathbb{N}$ be divisible by 12. Then there is an oriented graph G on n vertices with $\delta^0(G) = 5n/12 - 1$ which does not contain the square of a Hamilton cycle.*

Proof. Let G denote the oriented graph on n vertices whose vertex set consists of the sets A, B, C, D and E where $|A| = n/6+1$, $|B| = n/6-1$, $|C| = n/3$ and $|D| = |E| = n/6$. The edge set of G is obtained as follows: Add all possible edges from $A \cup B$ to C, from C to $D \cup E$, from D to $A \cup B$ and from E to $A \cup D$. Let B, C and D all induce tournaments that are as regular as possible (so $\delta^0(G[B]) = \delta^0(G[D]) = n/12 - 1$ and $\delta^0(G[C]) = n/6 - 1$). We add edges between A and B in such a way that every vertex in A sends and receives at least $n/12 - 1$ edges to and from B, and every vertex in B sends and receives at least $n/12$ edges to and from A. Similarly, we add edges between B and E in such a way that every vertex in B sends and receives $n/12$ edges to and from E, and every vertex in E sends and receives at least $n/12 - 1$ edges to and from B. A and E are both independent sets (see Figure 1). So $\delta^0(G) = 5n/12 - 1$.

Assume that G contains the square of a Hamilton cycle F. Since $|B| < |E|$, showing that F must visit B between any two visits of E would yield a contradiction. Thus, consider any vertex $e \in E$. Its predecessor c_1 on F lies in $B \cup C$, so without loss of generality we may assume that $c_1 \in C$. The predecessor c_2 of c_1 on F must lie in $N^-(e) \cap N^-(c_1) \subseteq B \cup C$. So without loss of generality we may assume that $c_2 \in C$. The predecessor c_3 of c_2 on F lies in $A \cup B \cup C$. Again we are done if $c_3 \in B$, so we assume that $c_3 \in A \cup C$. Since F visits all the vertices of G we must eventually arrive at a predecessor $a \in A$ whose successor c on F lies in C. But now the predecessor of a on F must lie in $N^-(c) \cap N^-(a) \subseteq B$, as required. $\qquad\square$

1.2. **Transitive triangle packings.** Given an (oriented) graph H, a *perfect H-packing* in an (oriented) graph G is a collection of vertex-disjoint copies of H which covers all the vertices in G. (Perfect H-packings are also referred to as H-*factors* or *perfect H-tilings*.) Perfect H-packings in graphs have been widely studied. Hajnal and Szemerédi [8] characterised the minimum degree which ensures a graph G contains a perfect K_r-packing. More recently, Kühn and Osthus [12] characterised, up to an additive constant, the minimum degree which ensures a graph G contains a perfect H-packing for an arbitrary graph H. Far less is known in the oriented graph case. Keevash and Sudakov [10] showed that any oriented graph G on n vertices with $\delta^0(G) \geq (1/2 - o(1))n$ contains a packing of cyclic triangles covering all but at most 3 vertices.

It is natural to ask for the minimum semidegree of an oriented graph which ensures a perfect packing of transitive triangles T_3. Note that if 3 divides $|G|$ then a necessary condition for an oriented graph G to contain a square of a Hamilton cycle is that G contains a perfect packing of transitive triangles. Let $\delta(G)$ denote the minimum degree
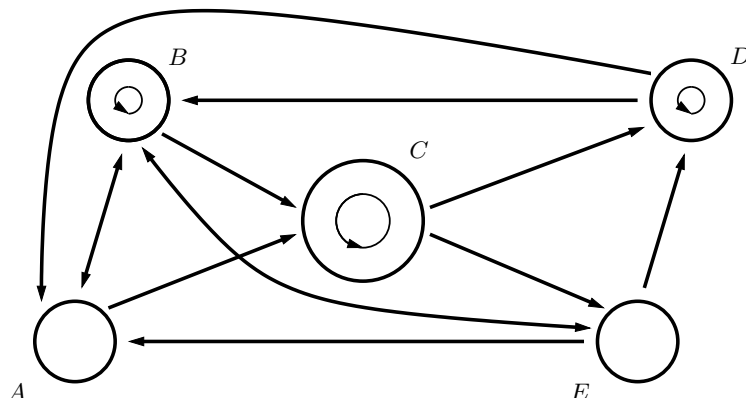
FIGURE 1. The oriented graph G from Proposition 3

of an oriented graph G (that is, the minimum number of edges incident to a vertex in G). The following proposition from [20] implies that a minimum semidegree as in Conjecture 2 ensures a perfect T_3-packing.

Proposition 4 (Yuster [20]). *Suppose G is an oriented graph whose order n is divisible by 3. If $\delta(G) \geq 5n/6$ then G contains a perfect T_3-packing.*

Proposition 4 is best possible in the sense that there are oriented graphs G whose order n is divisible by 3 and where $\delta(G) = (5n - 3)/6$ but which do not contain a perfect T_3-packing. (Indeed, consider the oriented graph G on $6m + 3$ vertices consisting of 3 vertex sets A, B and C where $|A| = |B| = m + 1$ and $|C| = 4m + 1$, and such that C induces a tournament, A sends out all possible edges to B, B sends out all possible edges to C and C sends out all possible edges to A. Then G does not contain a perfect T_3-packing since every copy of T_3 in G has at most one vertex in $A \cup B$.) However, when considering embeddings in oriented graphs, it seems that the more natural parameter to look at is the minimum semidegree. We believe that, in terms of minimum semidegree, one can improve on the bound given in Proposition 4.

Conjecture 5. *Suppose G is an oriented graph whose order n is divisible by 3. If $\delta^0(G) \geq 7n/18$ then G contains a perfect T_3-packing.*

If true, Conjecture 5 would characterise the minimum semidegree which ensures an oriented graph has a perfect T_3-packing.

Proposition 6. *Let $n \in \mathbb{N}$ be divisible by 18. Then there is an oriented graph G on n vertices with $\delta^0(G) = 7n/18 - 1$ which does not contain a perfect T_3-packing.*

Proof. Let G denote the oriented graph on n vertices whose vertex set consists of the sets A, B, C and D where $|A| = 2n/9 + 1$, $|B| = |C| = 2n/9$ and $|D| = n/3 - 1$ and whose edge set is obtained as follows: Add all possible edges from A to B, from B to C and from C to A. Let D induce a regular tournament. Partition D into two sets D' and D'' of sizes $n/6$ and $n/6 - 1$ respectively. Add all possible edges from D' to $B \cup C$, from A to D', from D'' to A and from $B \cup C$ to D'' (see Figure 2). It is easy to see that $\delta^0(G) = 7n/18 - 1$. Note that G does not have a perfect T_3-packing since every copy of T_3 in G must have at least one vertex in D. □
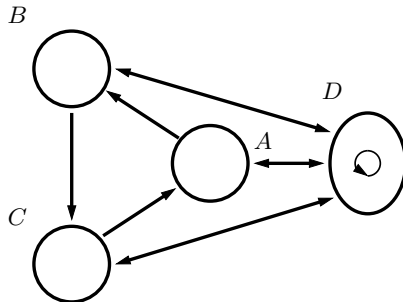
FIGURE 2. The oriented graph G from Proposition 6

1.3. **Packing transitive tournaments.** Let T_k denote the transitive tournament on k vertices. In light of Conjecture 5 we ask the following question.

Question 7. *What minimum semidegree condition ensures that an oriented graph contains a perfect T_k-packing?*

Recall that in the oriented graph G given in Proposition 6 the vertex set $A \cup B \cup C$ induces an oriented graph which does not contain a copy of T_3. This is the 'reason' why G does not contain a perfect T_3-packing. It would be of interest to establish whether the extremal examples, in terms of perfect T_k-packings, take a similar form. Thus, Question 7 is closely linked to the following question.

Question 8. *What minimum semidegree condition ensures that an oriented graph contains a copy of T_k?*

Valadkhan [19] has investigated this problem with respect to density conditions (a wider collection of problems of this nature are considered in [2]). It is easy to see that an oriented graph G on n vertices with $\delta^0(G) > n/3$ contains a copy of T_3 (and the blow-up of a cyclic triangle shows that this bound is best possible).

1.4. **Perfect packings and Ramsey numbers.** The *oriented tiling Ramsey number* $\overrightarrow{TR}(k)$ *of* k is the smallest integer n divisible by k such that any orientation of the complete graph K_n contains a perfect T_k-packing. Erdős (see [14]) proved the existence of these numbers. The following simple result gives a bound on the minimum degree which ensures an oriented graph G contains a perfect T_k-packing.

Proposition 9. *Suppose G is an oriented graph whose order n is divisible by k and such that $\delta(G) \geq (1 - \frac{1}{\overrightarrow{TR}(k)})n$. Then G contains a perfect T_k-packing.*

Sketch proof. Let $m := \overrightarrow{TR}(k)$. Consider the case when m divides n. By disregarding the orientations of the edges of G we obtain a graph G^* on n vertices with $\delta(G^*) \geq (1 - \frac{1}{m})n$. The Hajnal-Szemerédi theorem [8] implies that G^* has a perfect K_m-packing. By definition of m this implies that G has a perfect T_k-packing. If n is not divisible by m, we remove a number of vertex-disjoint copies of T_k from G until m divides $|G|$. We then proceed as before. $\square$

Note that $\overrightarrow{TR}(3) = 6$ so Proposition 9 implies Proposition 4. In view of Proposition 9 it is natural to seek good upper bounds on $\overrightarrow{TR}(k)$. The *oriented Ramsey number* $\overrightarrow{R}(k)$ *of* k is the smallest integer n such that any orientation of K_n contains a copy of T_k. The following proposition gives an upper bound on $\overrightarrow{TR}(k)$ in terms of oriented Ramsey numbers.

Proposition 10. *Given any* $k \in \mathbb{N}$, $\overrightarrow{TR}(k) \leq \overrightarrow{R}(2k-1) + (2k-1)\overrightarrow{R}(k)$.

Proof. We use the same trick as Caro used in [3]. Let n be the largest integer divisible by k such that $n \leq \overrightarrow{R}(2k-1) + (2k-1)\overrightarrow{R}(k)$ and ℓ the largest integer divisible by k which satisfies $\ell \leq \overrightarrow{R}(k)$. Consider any orientation $\overrightarrow{K}$ of K_n. By definition of n, $\overrightarrow{K}$ contains ℓ vertex-disjoint copies of T_{2k-1}. We can cover all but ℓ of the remaining vertices of $\overrightarrow{K}$ with vertex-disjoint copies of T_k. Each of the ℓ uncovered vertices x are paired off with one of our copies T'_{2k-1} of T_{2k-1}. Since x either sends out at least k edges to T'_{2k-1} in $\overrightarrow{K}$ or receives at least k edges from T'_{2k-1} in $\overrightarrow{K}$, we have that the oriented subgraph of $\overrightarrow{K}$ induced by $V(T'_{2k-1}) \cup \{x\}$ contains a perfect T_k-packing. Thus $\overrightarrow{K}$ contains a perfect T_k-packing. $\qquad\square$

The numbers $\overrightarrow{R}(k)$ are known for $k \leq 6$ (see [15, 16]). Sanchez-Flores [17] showed that $\overrightarrow{R}(7) \leq 54$ which by an induction argument implies that $\overrightarrow{R}(k) \leq 54 \cdot 2^{k-7}$ for $k \geq 7$ (this is the best known general upper bound on oriented Ramsey numbers). Note also that $\overrightarrow{R}(k) \leq R(k)$ where $R(k)$ denotes the Ramsey number of k.

Acknowledgement

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Andrew Treglown
School of Mathematics
University of Birmingham
Edgbaston
Birmingham
B15 2TT
UK

`treglowa@maths.bham.ac.uk`