Monochromatic triangles in three-coloured graphs

Andrew Treglown

Charles University, Prague

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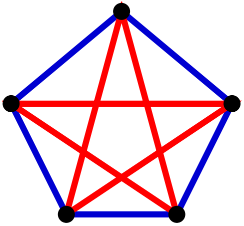
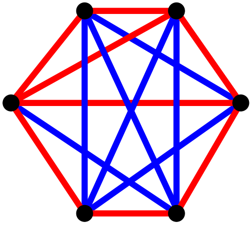
Joint work with James Cummings (Carnegie Mellon), Daniel Král' (Charles University), Florian Pfender (Colorado), Konrad Sperfeld (Universität Rostock) and Michael Young (Iowa State)

Ramsey theory

- Ramsey theory concerns questions of the following type: What is the smallest n such that K_n contains a monochromatic triangle whenever its edge set is 2-coloured?





- If $n = 5$, may have no monochromatic triangle. If $n \geq 6$, you must!

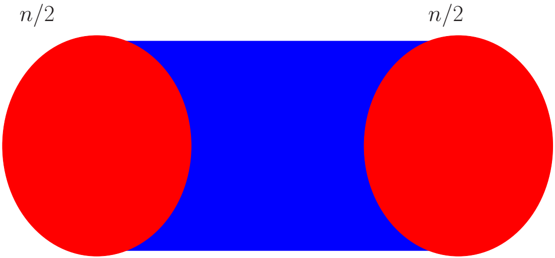
Ramsey multiplicity

- Ramsey's theorem $\forall\ k \in \mathbb{N}$ and any graph H, if n suff. large $\implies K_n$ contains a monochromatic H for any k-colouring
- It is natural therefore to ask how many monochromatic H must a k-coloured copy of K_n contain?
- Ramsey multiplicity $M_k(H, n) =$ minimum number of monochromatic H over all k-colourings of K_n.
- e.g. $M_2(K_3, 5) = 0$ and $M_2(K_3, 6) \geq 1$ (actually, $M_2(K_3, 6) = 2$).

- *How many* monochromatic triangles must a 2-coloured K_n contain (for $n \geq 6$)?

Theorem (Goodman 1959)

If K_n is 2-coloured $\implies$ at least $2\binom{n/2}{3}$ monochromatic triangles.



So $M_2(K_3, n) = 2\binom{n/2}{3}$.

Monochromatic triangles in 3-coloured graphs

- Goodman also asked for a 3-coloured analogue.
- Giraud (1976): $M_3(K_3, n) > 4\binom{n}{3}/115$ for large n.
- Sane and Walis (1988): $M_3(K_3, 17) = 5$.
  (Note that $R_3(K_3) = 17$ so $M_3(K_3, 16) = 0$.)

Theorem (Cummings, Král', Pfender, Sperfeld, T., Young 2012+)

If n large and K_n is 3-coloured $\implies$ at least $5\binom{n/5}{3} \approx 0.04\binom{n}{3}$ monochromatic triangles.

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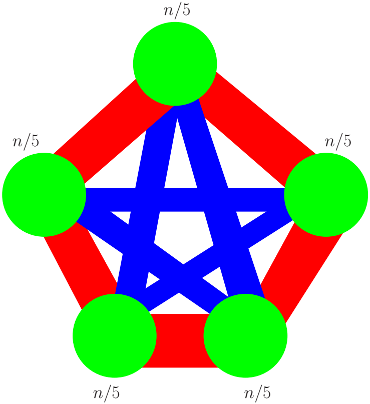
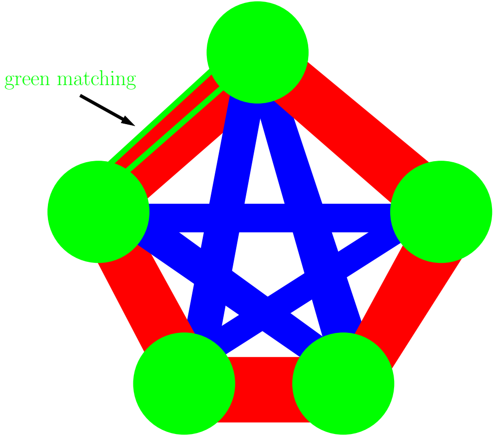


So $M_3(K_3, n) = 5\binom{n/5}{3}$ for large n.

- Notice that the extremal graph isn't unique.



- Let $\mathcal{G}_n$ denote class of all such 3-coloured K_n.

- We actually prove a stronger result.

Theorem (Cummings, Král', Pfender, Sperfeld, T., Young 2012+)

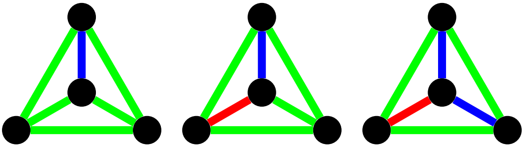
Suppose n sufficiently large and G is 3-coloured K_n containing minimum number of monochromatic K_3 $\implies$ G is member of $\mathcal{G}_n$

- So we have characterised all the extremal examples.

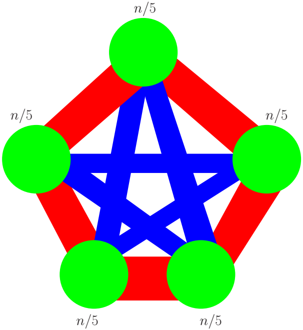
Theorem (CKPSTY)

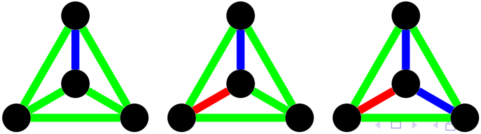
Suppose n large and G is 3-coloured K_n containing min. number of mono. $K_3 \implies G \in \mathcal{G}_n$

Consider the following family of 3-coloured graphs $\mathcal{H}$



- Note that no graph in $\mathcal{G}_n$ contains an element of $\mathcal{H}$ as a subgraph.





- Using Razborov's method of flag algebras we prove the following.

Proposition

$\forall\ \varepsilon > 0$, *if* n *large and* G *is* 3*-coloured copy of* K_n *then:*

(i) G *contains* $\geq (0.04 - \varepsilon)\binom{n}{3}$ *mono.* K_3

(ii) *If* G *contains* $\leq 0.04\binom{n}{3}$ *mono.* $K_3 \implies$ G *contains* $\leq \varepsilon\binom{n}{4}$ *copies of graphs in* $\mathcal{H}$.

- So if G = 3-coloured K_n with minimum number of mono. K_3 then G satisfies (i) and (ii).

(i) G contains $\geq (0.04 - \varepsilon)\binom{n}{3}$ mono. K_3

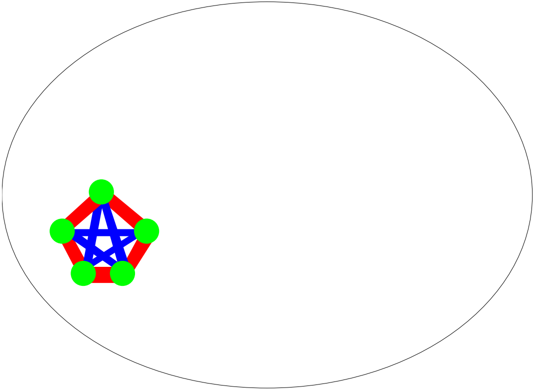
(ii) G contains $\leq \varepsilon\binom{n}{4}$ copies of graphs in $\mathcal{H}$.

Let $n_1 \ll n$.

- Call a set V of n_1 vertices standard if
  (1) $G[V]$ contains $\leq (0.04 + o(1))\binom{n_1}{3}$ mono. K_3;
  (2) $G[V]$ contains *no* element of $\mathcal{H}$.

With high probability a random sample of n_1 vertices is standard.

A standard subgraph 'looks' like an element of $\mathcal{G}_{n_1}$.



Let $n_2 \ll n_1$.

We can find a set V of n_2 vertices such that
(1) $G[V]$ looks like an element of $\mathcal{G}_{n_2}$;
(2) $G[V \cup \{u, v\}]$ looks like an element of $\mathcal{G}_{n_2+2}$ for *almost all* pairs of vertices u, v.

$\implies$ G 'close' to an element of $\mathcal{G}_n$.

$\implies$ G is an element of $\mathcal{G}_n$.

Open problems

- Lower the bound on n in our theorem.
  (Note the result can't hold for all n though.)

- Prove analogous results for more colours ($k \geq 4$).