On solution-free sets of integers

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(Joint work with Robert Hancock)
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Solution-free sets: Introduction

Let $[n] := \{1, \ldots, n\}$ and let $\mathcal{L}$ be $a_1x_1 + \cdots + a_kx_k = b$ where $a_1, \ldots, a_k, b \in \mathbb{Z}$.

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Examples

1. Sum-free sets (sets avoiding solutions to $x + y = z$)
2. Sidon sets (sets avoiding solutions to $x + y = z + t$)
3. Progression-free sets ($x + y = 2z$)

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Definitions:

1. $\mathcal{L}$ is translation-invariant if $\sum a_i = b = 0$.
2. A subset $A \subseteq [n]$ is $\mathcal{L}$-free if it does not contain any 'non-trivial' solutions to $\mathcal{L}$.
3. A subset $A \subseteq [n]$ is a maximal $\mathcal{L}$-free set if it is $\mathcal{L}$-free, and if the addition of any further $x \in [n] \setminus A$ would make it no longer $\mathcal{L}$-free.

Solution-free sets: Introduction

Fundamental Questions

- **Q1:** What is the size of the largest $\mathcal{L}$-free subset of $[n]$?
- **Q2:** How many $\mathcal{L}$-free subsets of $[n]$ are there?
- **Q3:** How many maximal $\mathcal{L}$-free subsets of $[n]$ are there?

Q1: What is the size of the largest $\mathcal{L}$-free subset of $[n]$?

Let $\mu_{\mathcal{L}}(n)$ be the size of the largest $\mathcal{L}$-free subset of $[n]$.

$\mathcal{L}$	$\mu_{\mathcal{L}}(n)$	Comment
$x + y = z$	$\lceil n/2 \rceil$	odds or interval
$x + y = 2z$	$o(n)$	Roth's theorem (1953)
$p(x + y) = rz$, $r > 2p$	$n - \lfloor 2n/r \rfloor$	union (Hegarty 2007)

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Generally...

$\mathcal{L}$	$\mu_{\mathcal{L}}(n)$
translation-invariant	$o(n)$
not translation-invariant	$\Omega(n)$

Q1: What is the size of the largest $\mathcal{L}$-free subset of $[n]$?

Theorem (Hancock, T. 2015+)

Let $\mathcal{L}$ be $px + qy = z$ where $p \geq q$ and $p \geq 2, p, q \in \mathbb{N}$. If n is sufficiently large then $\mu_{\mathcal{L}}(n) = n - \lfloor n/(p+q) \rfloor$.

- More recently, we have determined $\mu_{\mathcal{L}}(n)$ for a range of different equations $\mathcal{L}$ of the form $px + qy = rz$ where $p \geq q \geq r$.
- In each case, the extremal examples are 'intervals' or 'congruency classes'.

Q2: How many $\mathcal{L}$-free subsets of $[n]$ are there?

Let $f(n, \mathcal{L})$ be the number of $\mathcal{L}$-free subsets of $[n]$.
Clearly for any $\mathcal{L}$, we have $f(n, \mathcal{L}) \geq 2^{\mu_{\mathcal{L}}(n)}$.

Conjecture (Cameron-Erdős 1990)

Let $\mathcal{L}$ be $x + y = z$. Then $f(n, \mathcal{L}) = \Theta(2^{n/2})$.

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Theorem (Green, Sapozhenko 2003)

Let $\mathcal{L}$ be $x + y = z$. Then $\exists\, C_1, C_2$ s.t. given any $n \equiv i$ mod 2,
$f(n, \mathcal{L}) = (C_i + o(1))2^{n/2}$.

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Observation (Cameron-Erdős 1990)

Let $\mathcal{L}$ be translation-invariant. Then it is not true that $f(n, \mathcal{L}) = \Theta(2^{\mu_{\mathcal{L}}(n)})$.

Q2: How many $\mathcal{L}$-free subsets of $[n]$ are there?

Theorem (Green 2005)

Let $\mathcal{L}$ be $a_1x_1 + \cdots + a_kx_k = 0$ where $a_1, \ldots, a_k \in \mathbb{Z}$. Then $f(n, \mathcal{L}) = 2^{\mu_{\mathcal{L}}(n)+o(n)}$ (where $o(n)$ depends on $\mathcal{L}$).

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Theorem (Hancock, T. 2015+)

Fix $p, q \in \mathbb{N}$ where (i) $q \geq 2$ and $p > q(3q-2)/(2q-2)$ or (ii) $q = 1$ and $p \geq 3$. Let $\mathcal{L}$ be $px + qy = z$. Then $f(n, \mathcal{L}) = \Theta(2^{\mu_{\mathcal{L}}(n)})$.

Q3: How many maximal $\mathcal{L}$-free subsets of $[n]$ are there?

Let $f_{\max}(n, \mathcal{L})$ be the number of maximal $\mathcal{L}$-free subsets of $[n]$.

Question (Cameron-Erdős 1999)

Let $\mathcal{L}$ be $x + y = z$. Is it true that $f_{\max}(n, \mathcal{L}) = o(f(n, \mathcal{L}))$ or even $f_{\max}(n, \mathcal{L}) \leq f(n, \mathcal{L})/2^{\varepsilon n}$ for some constant $\varepsilon > 0$?

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Theorem (Łuczak-Schoen 2001)

Let $\mathcal{L}$ be $x+y=z$. Then $f_{\max}(n,\mathcal{L}) \leq 2^{n/2-2^{-28}n}$.

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Theorem (Balogh-Liu-Sharifzadeh-Treglown 2015)

Let $\mathcal{L}$ be $x + y = z$. For each $1 \leq i \leq 4$, there is a constant C_i s.t. given any $n \equiv i \bmod 4$, $f_{\max}(n, \mathcal{L}) = (C_i + o(1))2^{n/4}$.

Q3: How many maximal: An initial upper bound

Definition

- $\mathcal{L}$-triple: A solution to $\mathcal{L}$ when $\mathcal{L}$ is in three variables.
- $\mathcal{M}_{\mathcal{L}}(n)$: The set of $x \in [n]$ s.t. x does not lie in any $\mathcal{L}$-triple in $[n]$.
- $\mu^*_{\mathcal{L}}(n) := |\mathcal{M}_{\mathcal{L}}(n)|$.

Theorem (Hancock, T. 2015+)

Let $\mathcal{L}$ be $px + qy = rz$ where $p, q, r \in \mathbb{Z}$. Then $f_{\max}(n, \mathcal{L}) \leq 3^{(\mu_{\mathcal{L}}(n) - \mu^*_{\mathcal{L}}(n))/3 + o(n)}$.

Q3: How many maximal: Tools for upper bounds

Container lemma (Green 2005)

Let $\mathcal{L}$ be $px + qy = rz$ where $p, q, r \in \mathbb{Z}$.
There exists a family $\mathcal{F}$ of subsets of $[n]$ s.t.

(i) $\forall F \in \mathcal{F}$, $|F| \leq \mu_{\mathcal{L}}(n) + o(n)$ and F contains $\leq o(n^2)$ $\mathcal{L}$-triples; (F are 'containers' and are 'almost $\mathcal{L}$-free sets'.)

(ii) If $S \subseteq [n]$ $\mathcal{L}$-free, then $S \in F$ for some $F \in \mathcal{F}$; (Every $\mathcal{L}$-free set is in a container.)

(iii) $|\mathcal{F}| = 2^{o(n)}$. (There aren't many containers.)

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Removal lemma (Green 2005)

If $A \subseteq [n]$ contains $o(n^2)$ $\mathcal{L}$-triples, then $\exists\, B, C$ s.t. $A = B \cup C$ where B is $\mathcal{L}$-free and $|C| = o(n)$. (Every container is an $\mathcal{L}$-free set plus a 'very small' set.)

Q3: How many maximal: Tools for upper bounds

Link graphs

Given two subsets $B, S \subseteq [n]$, the link graph $L_S[B]$ of S on B is defined to have

- vertex set B;
- an edge between x and y if $\exists\, z \in S$ s.t. $\{x, y, z\}$ is an $\mathcal{L}$-triple;
- a loop at x if $\exists\, z, z' \in S$ s.t. $\{x, x, z\}$ or $\{x, z, z'\}$ is an $\mathcal{L}$-triple.

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Lemma

Suppose that B, S are disjoint $\mathcal{L}$-free subsets of $[n]$ and suppose $I \subseteq B$. If $S \cup I$ is a maximal $\mathcal{L}$-free subset of $[n]$, then I is a maximal independent set in $L_S[B]$.

Q3: How many maximal: Tools for upper bounds

Bounds on no. maximal independent sets:

Moon-Moser (1965) MIS$(G) \leq 3^{n/3}$.

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Let $\mathcal{L}$ be $px + qy = z$ where $p \geq q \geq 2$ are integers s.t. $p \leq q^2 - q$ and $\gcd(p, q) = q$. Then $f_{\max}(n, \mathcal{L}) \leq 2^{(\mu_{\mathcal{L}}(n) - \mu^*_{\mathcal{L}}(n))/2 + o(n)}$.

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Theorem (Hancock, T. 2016++)

Let $\mathcal{L}$ be $qx + qy = z$ where $q \geq 2$ is an integer. Then $f_{\max}(n, \mathcal{L}) = 2^{n/2q + o(n)}$.

Open problems

- Give an asymptotic formula for $f_{\max}(n, \mathcal{L})$ for all linear $\mathcal{L}$!
- What about abelian groups? (Questions 1 and 2 have be resolved for sum-free sets in abelian groups.)