A sharp bound on the number of maximal sum-free sets

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Abstract

Cameron and Erdős [7] asked whether the number of *maximal* sum-free sets in $\{1,\dots,n\}$ is much smaller than the number of sum-free sets. In the same paper they gave a lower bound of $2^{\lfloor n/4 \rfloor}$ for the number of maximal sum-free sets. We prove the following: For each $1 \leq i \leq 4$, there is a constant C_i such that, given any $n \equiv i \bmod 4$, $\{1,\dots,n\}$ contains $(C_i + o(1))2^{n/4}$ maximal sum-free sets. Our proof makes use of container and removal lemmas of Green [9,10], a structural result of Deshouillers, Freiman, Sós and Temkin [8] and a recent bound on the number of subsets of integers with small sumset by Green and Morris [11].

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1 Introduction

A triple x, y, z is a *Schur triple* if $x+y=z$ (note x, y and z may not necessarily be distinct). A set S is *sum-free* if S does not contain a Schur triple. Let $[n] := \{1, \dots, n\}$. We say that $S \subseteq [n]$ is a *maximal sum-free subset of* $[n]$ if it is sum-free and it is not properly contained in another sum-free subset of $[n]$. Let $f(n)$ denote the number of sum-free subsets of $[n]$ and $f_{\max}(n)$ denote the number of maximal sum-free subsets of $[n]$. The study of sum-free sets of integers has a rich history. Clearly, any set of odd integers and any subset of $\{\lfloor n/2 \rfloor + 1, \dots, n\}$ is a sum-free set, hence $f(n) \geq 2^{n/2}$. Cameron and Erdős [6] conjectured that $f(n) = O(2^{n/2})$. This conjecture was proven independently by Green [9] and Sapozhenko [15]. In fact, they showed that there are constants C_1 and C_2 such that $f(n) = (C_i + o(1))2^{n/2}$ for all $n \equiv i \bmod 2$.

In a second paper, Cameron and Erdős [7] observed that $f_{\max}(n) \geq 2^{\lfloor n/4 \rfloor}$. Noting that all the sum-free subsets of $[n]$ described above lie in just two maximal sum-free sets, they asked whether $f_{\max}(n) = o(f(n))$ or even $f_{\max}(n) \leq f(n)/2^{\varepsilon n}$ for some constant $\varepsilon > 0$. Łuczak and Schoen [13] answered this question in the affirmative, showing that $f_{\max}(n) \leq 2^{n/2 - 2^{-28}n}$ for sufficiently large n. Later, Wolfovitz [17] proved that $f_{\max}(n) \leq 2^{3n/8 + o(n)}$. More recently, the authors [2] proved that the lower bound is essentially tight, proving that $f_{\max}(n) = 2^{(1/4+o(1))n}$. In [3] we give the following exact solution to the problem.

Theorem 1.1 *For each $1 \leq i \leq 4$, there is a constant C_i such that, given any $n \equiv i \bmod 4$, $[n]$ contains $(C_i + o(1))2^{n/4}$ maximal sum-free sets.*

The proof draws on a number of ideas from [2]. In particular, as in [2] we make use of 'container' and 'removal' lemmas of Green [9,10] as well as a result of Deshouillers, Freiman, Sós and Temkin [8] on the structure of sum-free sets. In order to avoid over-counting the number of maximal sum-free subsets of $[n]$, our proof also develops a number of new ideas, thereby making the argument substantially more involved. We use a bound on the number of subsets of integers with small sumset by Green and Morris [11] as well as several new bounds on the number of maximal independent sets in various graphs. Further, the proof provides information about the typical structure of the maximal sum-free subsets of $[n]$. Indeed, we show that almost all of the maximal sum-free subsets of $[n]$ look like one of two particular extremal constructions (see Section 2.3 for more details).

2 Background and an overview of the proof of Theorem 1.1

2.1 Independence and container theorems

An exciting recent development in combinatorics and related areas has been the emergence of 'independence' as a unifying concept. To be more precise, let V be a set and $\mathcal{E}$ a collection of subsets of V. We say that a subset I of V is an *independent set* if I does not contain any element of $\mathcal{E}$ as a subset. For example, if $V := [n]$ and $\mathcal{E}$ is the collection of all Schur triples in $[n]$ then an independent set I is simply a sum-free set. It is often helpful to think of $(V, \mathcal{E})$ as a hypergraph with vertex set V and edge set $\mathcal{E}$; thus an independent set I corresponds to an independent set in the hypergraph.

So-called 'container results' have emerged as a powerful tool for attacking many problems that concern counting independent sets. Roughly speaking, container results state that the independent sets of a given hypergraph H lie only in a 'small' number of subsets of the vertex set of H (referred to as *containers*), where each of these containers is an 'almost independent set'. Balogh, Morris and Samotij [4] and independently Saxton and Thomason [16], proved general container theorems for hypergraphs whose edge distribution satisfies certain boundedness conditions.

In the proof of Theorem 1.1 we apply the following container theorem of Green [9].

Lemma 2.1 (Proposition 6 in [9]) *There exists a family $\mathcal{F}$ of subsets of $[n]$ with the following properties.*

(i) Every member of $\mathcal{F}$ has at most $o(n^2)$ Schur triples.
(ii) If $S \subseteq [n]$ is sum-free, then S is contained in some member of $\mathcal{F}$.
(iii) $|\mathcal{F}| = 2^{o(n)}$.
(iv) Every member of $\mathcal{F}$ has size at most $(1/2 + o(1))n$.

We refer to the sets in $\mathcal{F}$ as *containers*.

In [2] we used Lemma 2.1 to prove that $f_{\max}(n) = 2^{(1+o(1))n/4}$. Indeed, we showed that every $F \in \mathcal{F}$ contains at most $2^{(1+o(1))n/4}$ maximal sum-free subsets of $[n]$ which by (ii) and (iii) yields the desired result. To obtain an exact bound on $f_{\max}(n)$ it is not sufficient to give a tight general bound on the number of maximal sum-free subsets of $[n]$ that lie in a container $F \in \mathcal{F}$. Indeed, such an $F \in \mathcal{F}$ could contain $O(2^{n/4})$ maximal sum-free subsets of $[n]$, and thus together with (iii) this still gives an error term in the exponent. In general, since containers may overlap, applications of container results may lead to 'over-counting'.

We therefore need to count the number of maximal sum-free subsets of $[n]$ in a more refined way. To explain our method, we first need to describe the constructions which imply that $f_{\max}(n) \geq 2^{\lfloor n/4 \rfloor}$.

2.2 Lower bound constructions

The following construction of Cameron and Erdős [7] implies that $f_{\max}(n) \geq 2^{\lfloor n/4 \rfloor}$. Let $n \in \mathbb{N}$ and let $m = n$ or $m = n-1$, whichever is even. Let S consist of m together with precisely one number from each pair $\{x, m-x\}$ for odd $x < m/2$. Then S is sum-free. Moreover, although S may not be maximal, no further odd numbers less than m can be added, so distinct S lie in distinct maximal sum-free subsets of $[n]$.

The following construction from [2] also yields the same lower bound on $f_{\max}(n)$. Suppose that $4|n$ and set $I_1 := \{n/2+1, \dots, 3n/4\}$ and $I_2 := \{3n/4+1, \dots, n\}$. First choose the element $n/4$ and a set $S' \subseteq I_2$. Then for every $x \in I_2 \setminus S'$, choose $x - n/4 \in I_1$. The resulting set S is sum-free but may not be maximal. However, no further element in I_2 can be added, thus distinct S lie in distinct maximal sum-free sets in $[n]$. There are $2^{|I_2|} = 2^{n/4}$ ways to choose S.

2.3 Counting maximal sum-free sets

The following result provides structural information about the containers $F \in \mathcal{F}$. Lemma 2.2 is implicitly stated in [2] and was essentially proven in [9]. It is an immediate consequence of a result of Deshouillers, Freiman, Sós and Temkin [8] on the structure of sum-free sets and a removal lemma of Green [10]. Here O denotes the set of odd numbers in $[n]$.

Lemma 2.2 *If $F \subseteq [n]$ has $o(n^2)$ Schur triples then either*

(a) $|F| \leq 0.47n$;

or one of the following holds for some $-o(1) \leq \gamma = \gamma(n) \leq 0.03$:

(b) $|F| = \left(\frac{1}{2} - \gamma\right) n$ and $F = A \cup B$ where $|A| = o(n)$ and $B \subseteq [(1/2-\gamma)n, n]$ is sum-free;

(c) $|F| = \left(\frac{1}{2} - \gamma\right) n$ and $F = A \cup B$ where $|A| = o(n)$ and $B \subseteq O$.

The crucial idea in the proof of Theorem 1.1 is that we show 'most' of the maximal sum-free subsets of $[n]$ 'look like' the examples given in Section 2.2: We first show that containers of type (a) house only a small (at most $2^{0.249n}$) number of maximal sum-free subsets of $[n]$. For type (b) containers we split the argument into two parts. More precisely, we count the number of maximal sum-free subsets S of $[n]$ with the property that (i) the smallest element of S is $n/4 \pm o(n)$ and (ii) the second smallest element of S is at least $n/2 - o(n)$. (For this we use a direct argument rather than counting such sets within the containers.) We then show that the number of maximal sum-free subsets of $[n]$ that lie in type (b) containers but that fail to satisfy one of (i) and (ii) is small ($o(2^{n/4})$). We use a similar idea for type (c) containers. Indeed, we show directly that the number of maximal sum-free subsets of $[n]$ that contain *at most* one even number is $O(2^{n/4})$. We then show that the number of maximal sum-free subsets of $[n]$ that lie in type (c) containers and which contain two or more even numbers is small ($o(2^{n/4})$).

2.4 Counting maximal independent sets in link graphs

In each of our cases, we give an upper bound on the number of maximal sum-free sets in a container by counting the number of maximal independent sets in various auxiliary graphs. (Similar techniques were used in [17,2], and in the graph setting in [5].)

More precisely, for any subsets $B, S \subseteq [n]$, let $L_S[B]$ be the *link graph of S on B* defined as follows. The vertex set of $L_S[B]$ is B. The edge set of $L_S[B]$ consists of the following two types of edges:

(i) Two vertices x and y are adjacent if there exists an element $z \in S$ such that $\{x, y, z\}$ forms a Schur triple;

(ii) There is a loop at a vertex x if $\{x, x, z\}$ forms a Schur triple for some $z \in S$ or if $\{x, z, z'\}$ forms a Schur triple for some $z, z' \in S$.

The following simple lemma from [2] is crucial for the proof of Theorem 1.1.

Lemma 2.3 ([2]) *Suppose that B and S are both sum-free subsets of $[n]$. If $I \subseteq B$ is such that $S \cup I$ is a maximal sum-free subset of $[n]$, then I is a maximal independent set in $G := L_S[B]$.*

Suppose that $F \in \mathcal{F}$ is a container of type (c). Then $F = A \cup B$ where $|A| = o(n)$ and $B \subseteq O$. Notice that every maximal sum-free subset of $[n]$ in F can be obtained in the following two steps:

(1) Choose a sum-free set $S \subseteq A$.

(2) Extend S in B, i.e. choose a set $R \subseteq B$ such that $R \cup S$ is a maximal sum-free subset of $[n]$.

Note that by Lemma 2.3, the number of possible extensions in (2) is bounded from above by the number of maximal independent sets in the link graph $L_S[B]$. So Lemma 2.3 can be used to give an upper bound on the number of sum-free subsets of $[n]$ in F. We use the same approach to count maximal sum-free sets in containers of type (a) and (b).

To count the number of extensions in (2) we make use of several bounds on the number of maximal independent sets in various graphs.

Theorem 2.4 *Suppose that G is a graph on n vertices. Let $\mathrm{MIS}(G)$ denote the number of maximal independent sets in G. Then the following bounds hold.*

(i) $\mathrm{MIS}(G) \leq 3^{n/3}$;

(ii) $\mathrm{MIS}(G) \leq 2^{n/2}$ *if G is additionally triangle-free;*

(iii) $\mathrm{MIS}(G) \leq 2^{n/2-k/25}$ *if G is triangle-free and contains k vertex-disjoint paths on 3 vertices.*

Theorem 2.4(i) was proven by Moon and Moser [14]; we use it count the number of maximal sum-free subsets of $[n]$ housed in type (a) containers. Theorem 2.4(ii) was proven in [12] and Theorem 2.4(iii) in [1]; both are used to count the number of maximal sum-free subsets of $[n]$ housed in type (a) containers. Another, more technical, bound on $\mathrm{MIS}(G)$ is proved in [3] and used in the case of type (c) containers.

2.5 The number of sets with small sumset

We also make use of the following result which bounds the number of sets with small sumset.

Lemma 2.5 (Green and Morris [11]) *Fix $\delta > 0$ and $R > 0$. Then the following hold for all integers $s \geq s_0(\delta, R)$. For any $D \in \mathbb{N}$ there are at most*

$$2^{\delta s}\binom{\frac{1}{2}Rs}{s}D^{\lfloor R+\delta \rfloor}$$

sets $S \subseteq [D]$ with $|S| = s$ and $|S+S| \leq R|S|$.

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