On a degree sequence analogue of Pósa's conjecture

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Abstract

A famous conjecture of Pósa from 1962 asserts that every graph on n vertices and with minimum degree at least $2n/3$ contains the square of a Hamilton cycle. The conjecture was proven for large graphs in 1996 by Komlós, Sárközy and Szemerédi [17]. We prove a degree sequence version of Pósa's conjecture: Given any $\eta > 0$, every graph G of sufficiently large order n contains the square of a Hamilton cycle if its degree sequence $d_1 \leq \cdots \leq d_n$ satisfies $d_i \geq (1/3 + \eta)n + i$ for all $i \leq n/3$. The degree sequence condition here is asymptotically best possible. Our approach uses a hybrid of the Regularity-Blow-up method and the Connecting-Absorbing method.

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1 Introduction

One of the most fundamental results in extremal graph theory is Dirac's theorem [10] which states that every graph G on $n \geq 3$ vertices with minimum

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degree $\delta(G)$ at least $n/2$ contains a Hamilton cycle. It is easy to see that the minimum degree condition here is best possible. The *square* of a Hamilton cycle C is obtained from C by adding an edge between every pair of vertices of distance two on C. A famous conjecture of Pósa from 1962 (see [11]) provides an analogue of Dirac's theorem for the square of a Hamilton cycle.

Conjecture 1.1 (Pósa [11]) *Let G be a graph on n vertices. If $\delta(G) \geq 2n/3$, then G contains the square of a Hamilton cycle.*

Again, it is easy to see that the minimum degree condition in Pósa's conjecture cannot be lowered. The conjecture was intensively studied in the 1990s (see e.g. [12,13,14,15,16]), culminating in its proof for large graphs G by Komlós, Sárközy and Szemerédi [17]. The proof applies Szemerédi's Regularity lemma and as such the graphs G considered are extremely large. More recently, the lower bound on the size of G in this result has been significantly lowered (see [7,20]).

Although the minimum degree condition is best possible in Dirac's theorem, this does not necessarily mean that one cannot significantly strengthen this result. Indeed, Ore [21] showed that a graph G of order $n \geq 3$ contains a Hamilton cycle if $d(x) + d(y) \geq n$ for all non-adjacent $x \neq y \in V(G)$. The following result of Pósa [22] provides a *degree sequence* condition that ensures Hamiltonicity.

Theorem 1.2 (Pósa [22]) *Let G be a graph on $n \geq 3$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$. If $d_i \geq i + 1$ for all $i < (n-1)/2$ and if additionally $d_{\lceil n/2 \rceil} \geq \lceil n/2 \rceil$ when n is odd, then G contains a Hamilton cycle.*

Notice that Theorem 1.2 is significantly stronger than Dirac's theorem as it allows for almost half of the vertices of G to have degree less than $n/2$. A theorem of Chvátal [8] generalises Theorem 1.2 by characterising all those degree sequences which ensure the existence of a Hamilton cycle in a graph: Suppose that the degrees of a graph G are $d_1 \leq \cdots \leq d_n$. If $n \geq 3$ and $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $i < n/2$ then G is Hamiltonian. Moreover, if $d_1 \leq \cdots \leq d_n$ is a degree sequence that does not satisfy this condition then there exists a non-Hamiltonian graph G whose degree sequence $d'_1 \leq \cdots \leq d'_n$ is such that $d'_i \geq d_i$ for all $1 \leq i \leq n$.

Recently there has been an interest in generalising Pósa's conjecture. An 'Ore-type' analogue of Pósa's conjecture has been proven for large graphs in [6,9]. A random version of Pósa's conjecture was proven by Kühn and Osthus in [19]. In [2], Allen, Böttcher and Hladký determined the minimum degree threshold that ensures a large graph contains a square cycle of a given

length. The problem of finding the square of a Hamilton cycle in a pseudorandom graph has recently been studied in [1]. Our focus here is to investigate degree sequence conditions that guarantee a graph contains the square of a Hamilton cycle. This problem was raised in the arXiv version of [3]. In [24] we prove the following degree sequence version of Pósa's conjecture.

Theorem 1.3 *Given any $\eta > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. If G is a graph on $n \geq n_0$ vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i \geq n/3 + i + \eta n \text{ for all } i \leq n/3,$$

then G contains the square of a Hamilton cycle.

Note that Theorem 1.3 allows for almost $n/3$ vertices in G to have degree substantially smaller than $2n/3$. However, it does not quite imply Pósa's conjecture for large graphs due to the term ηn. An example from the arXiv version of [3] shows that the term ηn in Theorem 1.3 cannot be replaced by $o(\sqrt{n})$ for every $i \leq n/3$. So in this sense Theorem 1.3 is close to best possible. (Extremal examples for Theorem 1.3 are discussed in more detail in Section 3.) We suspect though that the degrees in Theorem 1.3 can be capped at $2n/3$.

Conjecture 1.4 *Given any $\eta > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. If G is a graph on $n \geq n_0$ vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i \geq \min\{n/3 + i + \eta n, 2n/3\} \text{ for all } i,$$

then G contains the square of a Hamilton cycle.

It would be extremely interesting to establish an analogue of Chvátal's theorem for the square of a Hamilton cycle, i.e., to characterise those degree sequences which force the square of a Hamilton cycle.

The proof of Theorem 1.3 makes use of Szemerédi's Regularity lemma [25] and the Blow-up lemma [18]. In the next section, we give a more detailed overview.

2   Overview of the proof

Over the last few decades a number of powerful techniques have been developed for embedding problems in graphs. The Blow-up lemma [18], in combination with the Regularity lemma [25], has been used to resolve a number of long-standing open problems, including Pósa's conjecture for large graphs [17].

More recently, the so-called Connecting-Absorbing method developed by Rödl, Ruciński and Szemerédi [23] has also proven to be highly effective in tackling such embedding problems.

Typically, both these approaches have been applied to graphs with 'large' minimum degree. Our graph G in Theorem 1.3 may have minimum degree $(1/3 + o(1))n$. In particular, this is significantly smaller than the minimum degree threshold that forces the square of a Hamilton cycle in a graph (namely, $2n/3$). As we describe below, having vertices of relatively small degree makes the proof of Theorem 1.3 highly involved and rather delicate. Indeed, our proof draws on ideas from both the Regularity-Blow-up method and the Connecting-Absorbing method. Further, we also develop a number of new ideas in order to deal with these vertices of small degree.

2.1 *An approximate version of Pósa's conjecture*

In order to highlight some of the difficulties in the proof of Theorem 1.3, we first give a sketch of a proof of an approximate version of Pósa's conjecture. This is based on the proof of Pósa's conjecture for large graphs given in [20].

Let $0 < \varepsilon \ll \gamma \ll \eta$. Suppose that G is a sufficiently large graph on n vertices with $\delta(G) \geq (2/3 + \eta)n$. We wish to find the square of a Hamilton cycle in G. The proof splits into three main parts.

- **Step 1 (Absorbing path):** Find an 'absorbing' square path P_A in G such that $|P_A| \leq \gamma n$. P_A has the property that given *any* set $A \subseteq V(G) \setminus V(P_A)$ such that $|A| \leq 2\varepsilon n$, G contains a square path P with vertex set $V(P_A) \cup A$, where the first and last two vertices on P are the same as the first and last two vertices on P_A.

- **Step 2 (Reservoir set):** Let $G' := G \setminus V(P_A)$. Find a 'reservoir' set $\mathcal{R} \subseteq V(G')$ such that $|\mathcal{R}| \leq \varepsilon n$. $\mathcal{R}$ has the property that, given *arbitrary* disjoint ordered edges $ab, cd \in E(G)$, there are 'many' short square paths P in G so that: (i) The first two vertices on P are a, b respectively; (ii) The last two vertices on P are c, d respectively; (iii) $V(P) \setminus \{a, b, c, d\} \subseteq \mathcal{R}$.

- **Step 3 (Almost tiling with square paths):** Let $G'' := G' \setminus \mathcal{R}$. Find a collection $\mathcal{P}$ of a bounded number of vertex-disjoint square paths in G'' that together cover all but εn of the vertices in G''.

Assuming that $\delta(G) \geq (2/3 + \eta)n$, the proof of each of these three steps is not too involved. (Note though that the proof in [20] is more technical since there $\delta(G) \geq 2n/3$.)

After completing Steps 1–3, it is straightforward to find the square of a

Hamilton cycle in G. Indeed, suppose ab is the last edge on a square path P_1 from $\mathcal{P}$ and cd is the first edge on a square path P_2 from $\mathcal{P}$. Then Step 2 implies that we can 'go through' $\mathcal{R}$ to join P_1 and P_2 into a single square path in G. Repeating this process we can obtain a square cycle C in G that contains all the square paths from $\mathcal{P}$. Further, we may also incorporate the absorbing square path P_A into C. C now covers almost all the vertices of G. We then use P_A to absorb all the vertices from $V(G) \setminus V(C)$ into C to obtain the square of a Hamilton cycle.

2.2 A degree sequence version of Pósa's conjecture

Suppose that G is a sufficiently large graph on n vertices as in the statement of Theorem 1.3. A result of the second author [26] guarantees that G contains a collection of $\lfloor n/3 \rfloor$ vertex-disjoint triangles. Further, this result together with a simple application of the Regularity lemma implies that G in fact contains a collection $\mathcal{P}$ of a bounded number of vertex-disjoint square paths that together cover almost all of the vertices in G. So we can indeed prove an analogue of Step 3 in this setting. In particular, if we could find a reservoir set $\mathcal{R}$ as above, then certainly we would be able to join together the square paths in $\mathcal{P}$ through $\mathcal{R}$, to obtain an almost spanning square cycle C in G.

Suppose that $ab, cd \in E(G)$ and we wish to find a square path P in G between ab and cd. If $d_G(a), d_G(b) < n/2$ then it may be the case that a and b have no common neighbours. Then it is clearly impossible to find such a square path P between ab and cd (since ab does not lie in a single square path!). The degree sequence condition on G is such that almost $n/6$ vertices in G may have degree less than $n/2$. Therefore we cannot hope to find a reservoir set precisely as in Step 2 above.

We overcome this significant problem as follows. We first show that G contains a reservoir set $\mathcal{R}$ that can *only* be used to find a square path between pairs of edges $ab, cd \in E(G)$ of *large degree* (namely, at least $(2/3+\eta)n$). This turns out to be quite involved. In order to use $\mathcal{R}$ to join together the square paths $P \in \mathcal{P}$ into an almost spanning square cycle, we now require that the first and last two vertices on each such P have large degree.

To find such a collection of square paths $\mathcal{P}$ we first find a special collection $\mathcal{F}$ of so-called 'folded paths' in a reduced graph R of G. Roughly speaking, folded paths are a generalisation of the notion of a square path. Each such folded path $F \in \mathcal{F}$ will act as a 'guide' for embedding one of the paths $P \in \mathcal{P}$ into G. More precisely, there is a homomorphism from a square path P into a folded path F. In particular, the structure of F will ensure that the first and

last two vertices on P are 'mapped' to large degree vertices in G.

Given our new reservoir set $\mathcal{R}$ and collection of square paths $\mathcal{P}$, we again can obtain an almost spanning square cycle C in G. Further, if we could construct an absorbing square path P_A as in Step 1, we would be able to absorb the vertices in $V(G) \setminus V(C)$ to obtain the square of a Hamilton cycle. However, we were unable to construct such an absorbing square path, and do not believe there is a 'simple' way to construct one. (Though, one could construct such a square path P_A if one only requires P_A to absorb vertices of large degree.) Instead, our method now turns towards the Regularity-Blow-up approach.

Using the methods described, we can obtain an almost spanning square cycle in the reduced graph R of G. In fact, we obtain a much richer structure Z_ℓ in R called a 'triangle cycle'. Z_ℓ is a special 6-regular graph on 3ℓ vertices that contains the square of a Hamilton cycle. In particular, Z_ℓ contains a collection of vertex-disjoint triangles T_ℓ that together cover all the vertices in Z_ℓ. We then show that G contains an almost spanning structure $\mathcal{C}$ that looks like the 'blow-up' of Z_ℓ. More precisely, if $V(Z_\ell) = \{1, \ldots, 3\ell\}$ and $V_1, \ldots, V_{3\ell}$ are the corresponding clusters in G, then

- $V(\mathcal{C}) = V_1 \cup \cdots \cup V_{3\ell}$;
- $\mathcal{C}[V_i, V_j]$ is ε-regular whenever $ij \in E(Z_\ell)$;
- If ij is an edge in a triangle $T \in T_\ell$ then $\mathcal{C}[V_i, V_j]$ is ε-regular and every vertex in V_i has at least $\gamma|V_j|$ neighbours in V_j.

We call $\mathcal{C}$ a 'cycle structure'. The initial structure of $\mathcal{C}$ is such that it contains a spanning square cycle. However, since $\mathcal{C}$ is not necessarily spanning in G, this does not correspond to the square of a Hamilton cycle in G. We thus need to incorporate the 'exceptional vertices' of G into this cycle structure $\mathcal{C}$ in a balanced way so that at the end $\mathcal{C}$ (and hence G) contains the square of a Hamilton cycle. The rich structure of Z_ℓ and thus $\mathcal{C}$ is vital for this. Again particular care is needed when incorporating exceptional vertices of small degree into our cycle structure. This part of the proof builds on ideas used in [4,5].

3   Extremal examples for Theorem 1.3

In this section we describe examples which show that Theorem 1.3 is asymptotically best possible.

Given a fixed graph H, an *H-packing* in a graph G is a collection of vertex-disjoint copies of H in G. We say that an H-packing is *perfect* if it

contains $\lfloor |G|/|H| \rfloor$ copies of H in G, i.e. the maximum number. Observe that the square of a Hamilton cycle contains a perfect K_3-packing. The following proposition is a special case of Proposition 17 in [3]. It implies that one cannot replace ηn with -1 in Theorem 1.3.

Proposition 3.1 *Suppose that $n \in 3\mathbb{N}$, $k \in \mathbb{N}$ and $1 \leq k < n/3$. Then there exists a graph G on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i = \begin{cases} n/3 + k - 1 & \text{if} \quad 1 \leq i \leq k \\ 2n/3 & \text{if} \quad k+1 \leq i \leq n/3 + k \\ n - k - 1 & \text{if} \quad n/3 + k + 1 \leq i \leq n - k + 1 \\ n - 1 & \text{if} \quad n - k + 2 \leq i \leq n, \end{cases}$$

but such that G does not contain a perfect K_3-packing.

Proof. Construct G as follows. The vertex set of G is the union of disjoint sets V_1, V_2, A, B of sizes $n/3$, $2n/3 - 2k + 1$, $k - 1$, k respectively. Add all edges from $B \cup V_2 \cup A$ to V_1. Further, add all edges with both endpoints in $V_2 \cup A$. Add all possible edges between A and B.

Consider an arbitrary copy T of K_3 in G which contains $b \in B$. Since B is an independent set in G and there are no edges between B and V_2, we have that $V(T) \setminus \{b\} \subseteq A \cup V_1$. But V_1 is an independent set in G, so T contains at most one vertex in V_1 and hence at least one vertex in A. But since $|B| > |A|$ this implies that G does not contain a perfect K_3-packing. Furthermore, it is easy to check that G has our desired degree sequence. $\square$

Note that Proposition 3.1 shows that, if true, Conjecture 1.4 is close to best possible in the following sense: Given any $1 \leq k < n/3$, there is a graph G on n vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that (i) G does not contain the square of a Hamilton cycle and (ii) G satisfies the degree sequence condition in Conjecture 1.4 except for the terms $d_{k-\eta n}, \ldots, d_k$ which only 'just' fail to satisfy the desired condition.

At first sight, one might think that the ηn term in Theorem 1.3 is an artifact of our proof, but in fact it is a feature of the problem: indeed, it cannot be replaced by $o(\sqrt{n})$. This is shown by an example in Proposition 22 in the arXiv version of [3].

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