Perfect packings in graphs

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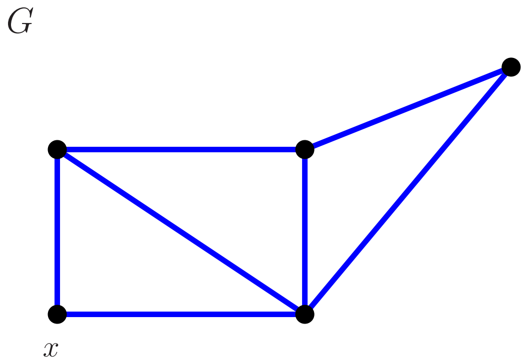
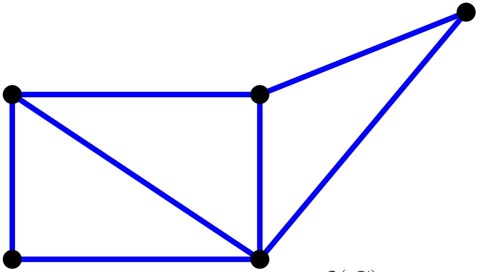
Joint work with Daniela Kühn and Deryk Osthus (University of Birmingham)



$d(x) = 2$

G

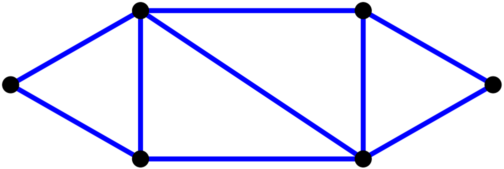


$\delta(G) = 2$

$d(G) = 14/5$

Motivation 1: Perfect matchings

- A matching in G is a collection of vertex-disjoint edges.
- A matching in G is perfect if it covers all vertices in G.



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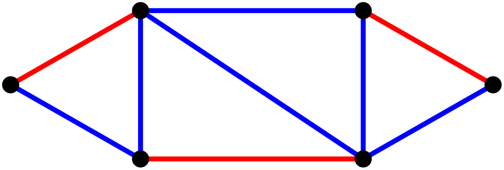

perfect matching

Perfect matchings in bipartite graphs

Theorem (Hall)

G bipartite graph with vertex classes X, Y

$$G \text{ has matching covering } X \iff \forall\, S \subseteq X,\ |N_G(S)| \geq |S|$$

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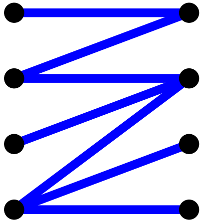

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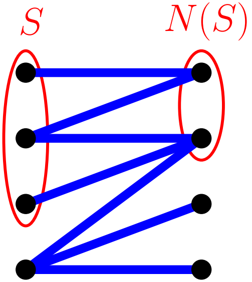
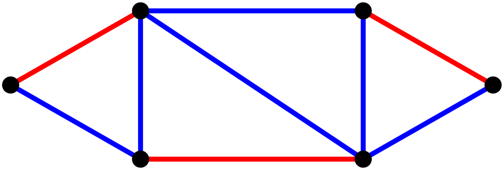


no perfect matching

Characterising graphs with perfect matchings

- Tutte's Theorem characterises all those graphs with perfect matchings.



Motivation 2: Finding a (small) graph H in G

- Vertex colouring of G: colour vertices so that adjacent vertices coloured differently.
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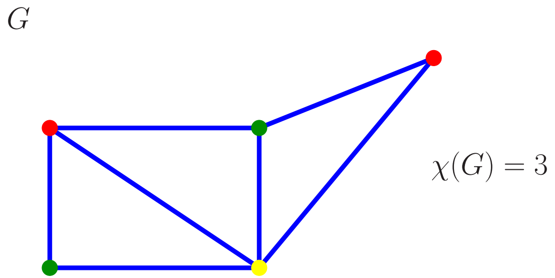
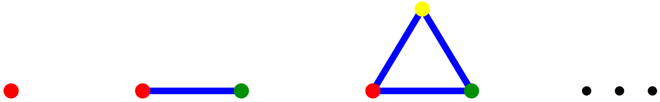


Complete graphs

K_r $\qquad$ $\chi(K_r) = r$



The Erdős-Stone Theorem

Theorem (Erdős, Stone '46)

Given $\eta > 0$, if G graph on sufficiently large n number of vertices and

$$e(G) \geq \left(1 - \frac{1}{\chi(H) - 1} + \eta\right) \frac{n^2}{2}$$

then $H \subseteq G$.

Corollary

Given $\eta > 0$, if G graph on sufficiently large n number of vertices and

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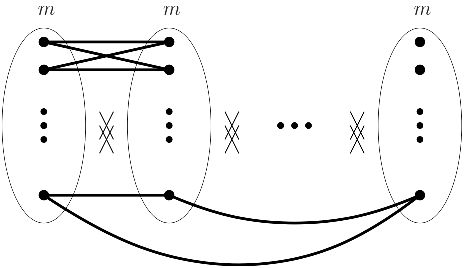
- Erdős-Stone Theorem best possible (up to error term).

H graph $\chi(H) = r$

G $\quad |G| = m(r-1)$



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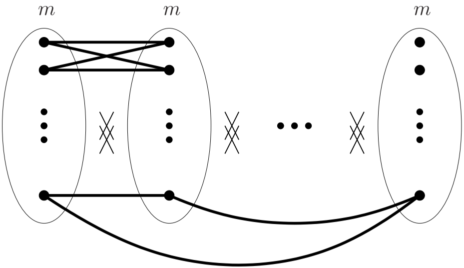

$\delta(G) = \left(1 - \frac{1}{r-1}\right)|G|$

$e(G) = \left(1 - \frac{1}{r-1}\right)\frac{|G|^2}{2}$

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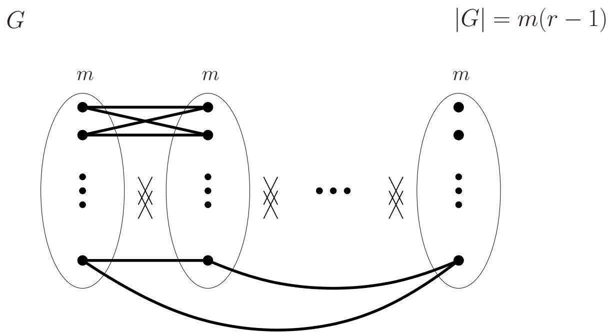

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no copy of H in G

Other types of degree condition

Ore-type degree conditions:
Consider the sum of the degrees of non-adjacent vertices.

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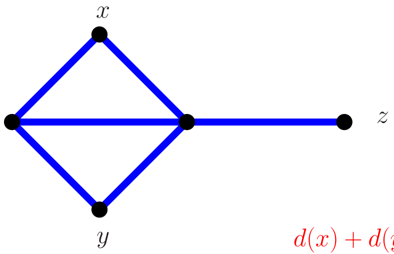

$d(x) + d(y) = 2 + 2 = 4$
$d(y) + d(z) = 2 + 1 = 3$

Properties of Ore-type conditions

- $\delta(G) \geq a \Rightarrow d(x) + d(y) \geq 2a \;\; \forall\, x, y \in V(G)$ s.t. $xy \notin E(G)$.
- $d(x) + d(y) \geq 2a \;\; \forall \ldots \Rightarrow d(G) \geq a$.

Corollary

Given $\eta > 0$, if G has sufficiently large order n and

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{\chi(H) - 1} + \eta\right) n \quad \forall \ldots$$

then $H \subseteq G$.

Properties of Ore-type conditions

- $\delta(G) \geq a \Rightarrow d(x)+d(y) \geq 2a \;\; \forall\, x, y \in V(G)$ s.t. $xy \notin E(G)$.
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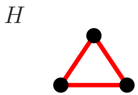

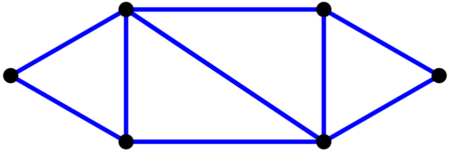

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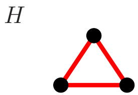

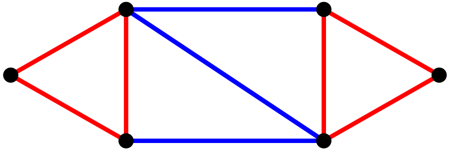

perfect H-packing

Remarks

- If $H = K_2$ then perfect H-packing $\iff$ perfect matching.
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Perfect K_r-packings

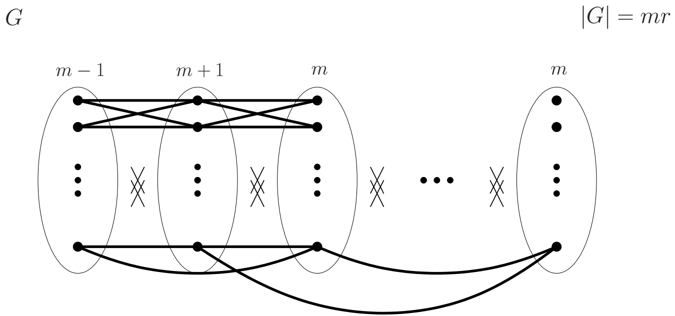
Theorem (Hajnal, Szemerédi '70)

G graph, $|G| = n$ where $r|n$ and

$$\delta(G) \geq \left(1 - \frac{1}{r}\right) n$$

$\Rightarrow$ G contains a perfect K_r-packing.

- Hajnal-Szemerédi Theorem best possible.



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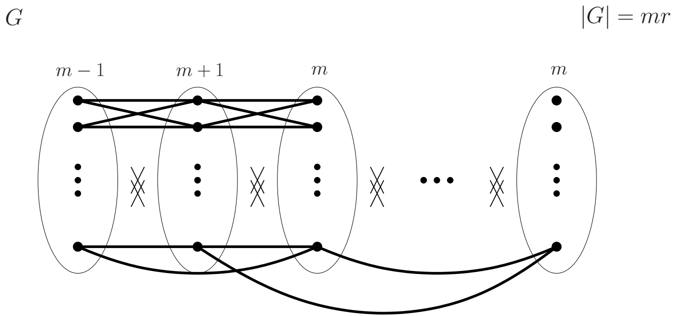

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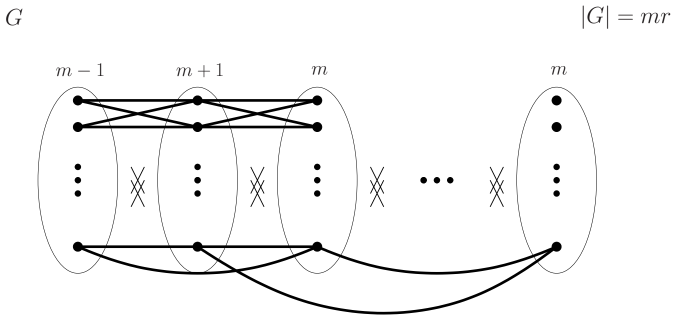

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no perfect K_r-packing

perfect H-packings for arbitrary H

- Given H, the critical chromatic number $\chi_{cr}(H)$ of H is

$$\chi_{cr}(H) := (\chi(H)-1)\frac{|H|}{|H|-\sigma(H)}$$

  where $\sigma(H)$ is the size of the smallest possible colour class in a $\chi(H)$-colouring of H.

- $\chi(H)-1 < \chi_{cr}(H) \leq \chi(H) \quad \forall\ H$

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Theorem (Kühn, Osthus)

$\forall\ H,\ \ \exists\ C$ s.t. if $|H|$ divides $|G|$ and

$$\delta(G) \geq \left(1 - \frac{1}{\chi^*(H)}\right)|G| + C$$

then G contains a perfect H-packing.
Here,

$$\chi^*(H) = \begin{cases} \chi(H) & \text{for some } H \text{ (including } K_r\text{);} \\ \chi_{cr}(H) & \text{otherwise.} \end{cases}$$

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Ore-type conditions

What Ore-type degree condition ensures a graph G contains a perfect H-packing?

Theorem (Kierstead, Kostochka '08)

G graph, $|G| = n$ where $r|n$ and

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{r}\right)n - 1 \quad \forall \ \ldots$$

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- Result implies Hajnal-Szemerédi Theorem.
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What about perfect H-packings for arbitrary H?
An example:

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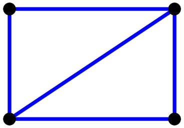

$\chi(H) = 3$ $\qquad$ $\chi_{cr}(H) = 8/3$

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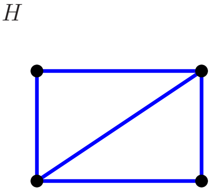

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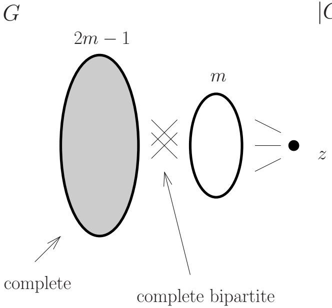
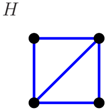
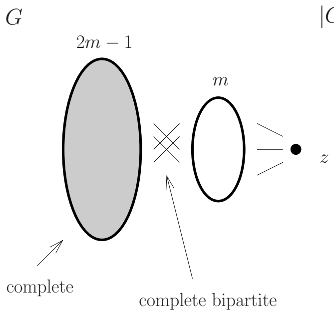


$|G| = 3m$





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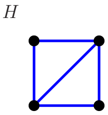
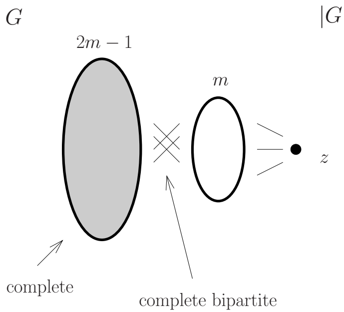


no perfect H-packing

$d(x) + d(y) \geq 4m - 2 = 2(1 - 1/\chi(H))|G| - 2 \quad \forall \ldots$

"Something else is going on!"



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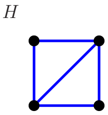

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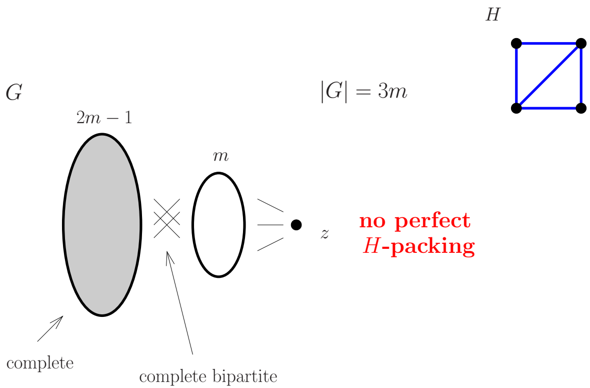

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We characterised, asymptotically, the Ore-type degree condition which ensures that a graph contains a perfect H-packing.

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