

# THE INDUCED SATURATION PROBLEM FOR POSETS

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ABSTRACT. For a fixed poset  $P$ , a family  $\mathcal{F}$  of subsets of  $[n]$  is induced  $P$ -saturated if  $\mathcal{F}$  does not contain an induced copy of  $P$ , but for every subset  $S$  of  $[n]$  such that  $S \notin \mathcal{F}$ ,  $P$  is an induced subposet of  $\mathcal{F} \cup \{S\}$ . The size of the smallest such family  $\mathcal{F}$  is denoted by  $\text{sat}^*(n, P)$ . Keszegh, Lemons, Martin, Pálvölgyi and Patkós [Journal of Combinatorial Theory Series A, 2021] proved that there is a dichotomy of behaviour for this parameter: given any poset  $P$ , either  $\text{sat}^*(n, P) = O(1)$  or  $\text{sat}^*(n, P) \geq \log_2 n$ . In this paper we improve this general result showing that either  $\text{sat}^*(n, P) = O(1)$  or  $\text{sat}^*(n, P) \geq \min\{2\sqrt{n}, n/2 + 1\}$ . Our proof makes use of a Turán-type result for digraphs.

Curiously, it remains open as to whether our result is essentially best possible or not. On the one hand, a conjecture of Ivan states that for the so-called diamond poset  $\diamond$  we have  $\text{sat}^*(n, \diamond) = \Theta(\sqrt{n})$ ; so if true this conjecture implies our result is tight up to a multiplicative constant. On the other hand, a conjecture of Keszegh, Lemons, Martin, Pálvölgyi and Patkós states that given any poset  $P$ , either  $\text{sat}^*(n, P) = O(1)$  or  $\text{sat}^*(n, P) \geq n + 1$ . We prove that this latter conjecture is true for a certain class of posets  $P$ .

## 1. INTRODUCTION

*Saturation* problems have been well studied in graph theory. A graph  $G$  is  $H$ -saturated if it does not contain a copy of the graph  $H$ , but adding any edge to  $G$  from its complement creates a copy of  $H$ . Turán's celebrated theorem [15] can be stated in the language of saturation: it determines the maximum number of edges in a  $K_r$ -saturated  $n$ -vertex graph. In contrast, Erdős, Hajnal and Moon [5] determined the minimum number of edges in a  $K_r$ -saturated  $n$ -vertex graph; see the survey [3] for further results in this direction.

In recent years there has been an emphasis on developing the theory of saturation for posets. Turán-type problems have been extensively studied in this setting (see, e.g., the survey [9]). In this paper we are interested in *minimum* saturation questions à la Erdős–Hajnal–Moon. In particular, we consider *induced saturation* problems.

All posets we consider will be (implicitly) viewed as finite collections of finite subsets of  $\mathbb{N}$ . In particular, we say that  $P$  is a *poset on*  $[p] := \{1, 2, \dots, p\}$  if  $P$  consists of subsets of  $[p]$ . Let  $P, Q$  be posets. A *poset homomorphism* from  $P$  to  $Q$  is a function  $\phi : P \rightarrow Q$  such that for every  $A, B \in P$ , if  $A \subseteq B$  then  $\phi(A) \subseteq \phi(B)$ . We say that  $P$  is a *subposet* of  $Q$  if there is an injective poset homomorphism from  $P$  to  $Q$ ; otherwise,  $Q$  is said to be  $P$ -free. Further we say  $P$  is an

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The main results of this paper were first announced in the conference abstract [7].

*induced subposet* of  $Q$  if there is an injective poset homomorphism  $\phi$  from  $P$  to  $Q$  such that for every  $A, B \in P$ ,  $\phi(A) \subseteq \phi(B)$  if and only if  $A \subseteq B$ ; otherwise,  $Q$  is said to be *induced  $P$ -free*.

For a fixed poset  $P$ , we say that a family  $\mathcal{F} \subseteq 2^{[n]}$  of subsets of  $[n]$  is  *$P$ -saturated* if  $\mathcal{F}$  is  $P$ -free, but for every subset  $S$  of  $[n]$  such that  $S \notin \mathcal{F}$ , then  $P$  is a subposet of  $\mathcal{F} \cup \{S\}$ . A family  $\mathcal{F} \subseteq 2^{[n]}$  of subsets of  $[n]$  is *induced  $P$ -saturated* if  $\mathcal{F}$  is induced  $P$ -free, but for every subset  $S$  of  $[n]$  such that  $S \notin \mathcal{F}$ , then  $P$  is an induced subposet of  $\mathcal{F} \cup \{S\}$ .

The study of minimum saturated posets was initiated by Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi and Patkós [8] in 2013. In their work the relevant parameter is  $\text{sat}(n, P)$ , which is defined to be the size of the smallest  $P$ -saturated family of subsets of  $[n]$ . See, e.g., [8, 12, 14] for various results on  $\text{sat}(n, P)$ .

The induced analogue of  $\text{sat}(n, P)$  – denoted by  $\text{sat}^*(n, P)$  – was first considered by Ferrara, Kay, Kramer, Martin, Reiniger, Smith and Sullivan [6]. Thus,  $\text{sat}^*(n, P)$  is defined to be the size of the smallest induced  $P$ -saturated family of subsets of  $[n]$ . The following result from [12] (and implicit in [6]) shows that the parameter  $\text{sat}^*(n, P)$  has a dichotomy of behaviour.

**Theorem 1.1.** [6, 12] *For any poset  $P$ , either there exists a constant  $K_P$  with  $\text{sat}^*(n, P) \leq K_P$  or  $\text{sat}^*(n, P) \geq \log_2 n$ , for all  $n \in \mathbb{N}$ .*

Probably the most important open problem in the area is to obtain a tight version of Theorem 1.1; that is, to replace the  $\log_2 n$  in Theorem 1.1 with a term that is as large as possible. In fact, Keszegh, Lemons, Martin, Pálvölgyi and Patkós [12] made the following conjecture in this direction.

**Conjecture 1.2.** [12] *For any poset  $P$ , either there exists a constant  $K_P$  with  $\text{sat}^*(n, P) \leq K_P$  or  $\text{sat}^*(n, P) \geq n + 1$ , for all  $n \in \mathbb{N}$ .*

Note that the lower bound of  $n + 1$  is rather natural here. For example, it is the size of the largest *chain* in  $2^{[n]}$  as well as the smallest possible size of the union of two consecutive ‘layers’ in  $2^{[n]}$ , namely the layer containing  $[n]$  and the layer containing all subsets of  $[n]$  of size exactly  $n - 1$ . Furthermore, such structures form minimum induced saturated families for the so-called fork poset  $\vee$ , i.e.,  $\text{sat}^*(n, \vee) = n + 1$  [6]; so the lower bound in Conjecture 1.2 cannot be increased. There are also no known examples of posets  $P$  for which  $\text{sat}^*(n, P) = \omega(n)$ .

In contrast, Ivan [11, Section 3] presented evidence that led her to conjecture a rather different picture for the *diamond* poset  $\diamond$  (see Figure 1 for the Hasse diagram of  $\diamond$ ).

**Conjecture 1.3.** [11]  *$\text{sat}^*(n, \diamond) = \Theta(\sqrt{n})$ .*

Our main result is the following improvement of Theorem 1.1.

**Theorem 1.4.** *For any poset  $P$ , either there exists a constant  $K_P$  with  $\text{sat}^*(n, P) \leq K_P$  or  $\text{sat}^*(n, P) \geq \min\{2\sqrt{n}, n/2 + 1\}$  for all  $n \in \mathbb{N}$ .*

Note that if  $n \geq 12$  then  $n/2 + 1 \geq 2\sqrt{n}$ . Thus, if Conjecture 1.3 is true, the lower bound in Theorem 1.4 would be tight up to a multiplicative constant.

On the other hand, we prove that Conjecture 1.2 does hold for a class of posets (that does not include  $\diamond$ ). Given  $p \in \mathbb{N}$  and a poset  $P$  on  $[p]$  we define the *dual*  $\overline{P}$  of  $P$  as  $\overline{P} := \{[p] \setminus F : F \in P\}$ . We say a poset  $P$  *has legs* if there are distinct elements  $L_1, L_2, H \in P$  such that  $L_1, L_2$  are incomparable,  $L_1, L_2 \subseteq H$  and for any other element  $A \in P \setminus \{L_1, L_2, H\}$  we have  $A \supseteq H$ . The elements  $L_1$  and  $L_2$  are called *legs* and  $H$  is called a *hip*.

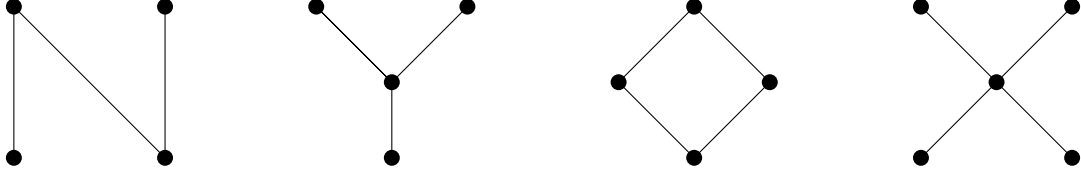


FIGURE 1. Hasse diagrams for the posets  $N$ ,  $Y$ ,  $\diamond$  and  $X$ .

**Theorem 1.5.** *Let  $P$  be a poset with legs and  $n \geq 3$ . Then  $\text{sat}^*(n, P) \geq n + 1$ . Moreover, if both  $P$  and  $\bar{P}$  have legs, then  $\text{sat}^*(n, P) \geq 2n + 2$ .*

Our results still leave both Conjecture 1.2 and Conjecture 1.3 open, and it is unclear to us which of these conjectures is true. However, if Conjecture 1.3 is true we believe it highly likely that there will be other posets  $P$  for which  $\text{sat}^*(n, P) = \Theta(\sqrt{n})$ .

It is also natural to seek exact results on  $\text{sat}^*(n, P)$ . However, despite there already being several papers concerning  $\text{sat}^*(n, P)$  [1, 4, 6, 10, 11, 12, 13], there are relatively few posets  $P$  for which  $\text{sat}^*(n, P)$  is known precisely (see, e.g., Table 1 in [12] for a summary of most of the known results). Our next result extends this limited pool of posets, determining  $\text{sat}^*(n, X)$  and  $\text{sat}^*(n, Y)$  (see Figure 1 for the Hasse diagrams of  $X$  and  $Y$ ).

**Theorem 1.6.** *Given any  $n \in \mathbb{N}$  with  $n \geq 3$ ,*

- (i)  $\text{sat}^*(n, Y) = n + 2$  and
- (ii)  $\text{sat}^*(n, X) = 2n + 2$ .

Note that  $\bar{X} = X$ , so Theorem 1.6(ii) easily follows via Theorem 1.5 and an extremal construction. An application of Theorem 1.5 to  $Y$  only yields that  $\text{sat}^*(n, Y) \geq n + 1$ , so we require an extra idea to obtain Theorem 1.6(i).

In [13] a trick was introduced which can be used to prove lower bounds on  $\text{sat}^*(n, P)$  for some posets  $P$ . The idea is to construct a certain auxiliary digraph  $D$  whose vertex set consists of the elements in an induced  $P$ -saturated family  $\mathcal{F}$ ; one then argues that how this digraph is defined forces  $D$  to contain many edges, which in turn forces a bound on the size of the vertex set of  $D$  (i.e., lower bounds  $|\mathcal{F}|$ ). This trick has been used to prove that  $\text{sat}^*(n, \diamond) \geq \sqrt{n}$  [13, Theorem 6] and  $\text{sat}^*(n, N) \geq \sqrt{n}$  [10, Proposition 4] (see Figure 1 for the Hasse diagram of  $N$ ).

Our proof of Theorem 1.4 utilises a variant of this digraph trick. In particular, by introducing an appropriate modification to the auxiliary digraph  $D$  used in [13], we are able to deduce certain Turán-type properties of  $D$ . Turán problems in digraphs are classical in extremal combinatorics and their study can be traced back to the work of Brown and Harary [2]. Here we prove a Turán-type result concerning *transitive cycles*.

Given  $k \geq 3$ , the *transitive cycle on  $k$  vertices*  $\overrightarrow{TC}_k$  is a digraph with vertex set  $[k]$  and every directed edge from  $i$  to  $i + 1$  for every  $i \in [k - 1]$ , as well as the directed edge from 1 to  $k$ . We establish an upper bound on the number of edges of a digraph not containing any transitive cycle.

**Theorem 1.7.** *Let  $n \in \mathbb{N}$  and let  $D$  be a digraph on  $n$  vertices. If  $D$  is  $\overrightarrow{TC}_k$ -free for all  $k \geq 3$ , then*

$$e(D) \leq \max \left\{ 2(n - 1), \left\lfloor \frac{n^2}{4} \right\rfloor \right\}.$$

Note that the bound in Theorem 1.7 is best possible. Indeed, consider the  $n$ -vertex digraph  $D$  with vertex classes  $A, B$  of size  $\lfloor n/2 \rfloor$  and  $\lceil n/2 \rceil$  respectively and all possible directed edges from  $A$

to  $B$ . So  $D$  has  $\lfloor n^2/4 \rfloor$  edges and contains no transitive cycle. Similarly, pick an arbitrary  $n$ -vertex tree  $T$  and let  $D'$  be the digraph with vertex set  $V(D') = V(T)$  and with an edge directed from  $i$  to  $j$  if and only if the unordered pair  $ij$  is an edge in  $T$ . So  $D'$  has  $2(n-1)$  edges and contains no transitive cycle.

In Section 3 we explain why exploiting the connection between the induced saturation problem for posets and the Turán problem for digraphs hits a natural barrier if we wish to further improve the lower bound in Theorem 1.4.

We conclude this introductory section by setting out notation that will be used in the rest of the paper.

**Notation.** Given two elements  $A, B$  of a poset  $\mathcal{F} \subseteq 2^{[n]}$  we say that  $A$  *dominates*  $B$  if  $B \subseteq A$ . We say  $A$  and  $B$  are *comparable* if one dominates the other; otherwise we say they are *incomparable*. An element  $F$  in  $\mathcal{F}$  is *maximal* if there is no other element in  $\mathcal{F}$  which dominates  $F$ . We define *minimal* analogously.

For a digraph  $D = (V(D), E(D))$ , we write  $\overrightarrow{xy}$  for the edge in  $D$  directed from the vertex  $x$  to the vertex  $y$ . We say that  $\overrightarrow{xy}$  is an *out-edge* of  $x$  and an *in-edge* of  $y$ . For brevity, we write  $e(D) := |E(D)|$ . Given a set  $U \subseteq V(D)$ , we denote the induced subgraph of  $D$  with vertex set  $U$  as  $D[U]$ . The *underlying graph* of  $D$  is the graph with vertex set  $V(D)$  whose edges are all (unordered) pairs  $\{x, y\}$  such that  $\overrightarrow{xy} \in E(D)$  or  $\overrightarrow{yx} \in E(D)$ .

Given  $k \geq 2$ , an *oriented path*  $v_1 \dots v_k$  from  $v_1$  to  $v_k$  consists of the edges  $\overrightarrow{v_i v_{i+1}}$  for every  $i \in [k-1]$ . Similarly, an *oriented cycle*  $v_1 \dots v_k$  consists of the edges  $\overrightarrow{v_i v_{i+1}}$  for every  $i \in [k-1]$  and  $\overrightarrow{v_k v_1}$ .

## 2. THE PROOFS

### 2.1. Proof of Theorem 1.7.

We proceed by induction on  $n$ . For the base case, it is easy to check that the statement of the theorem holds for  $n \leq 3$ . Next, we prove the inductive step.

Let  $D$  be a digraph on  $n \geq 4$  vertices which is  $\overrightarrow{TC}_k$ -free for all  $k \geq 3$ .

**Claim 2.1.** *If  $D$  contains an induced oriented cycle then  $e(D) \leq \max \left\{ 2(n-1), \left\lfloor \frac{n^2}{4} \right\rfloor \right\}$ .*

**Proof of the claim.** Suppose  $D$  contains an induced oriented cycle  $C$ . For every  $v \in V(D) \setminus V(C)$ , it is straightforward to check that, since  $D[V(C) \cup \{v\}]$  contains no transitive cycle, then

- (i) there is at most one in-edge of  $v$  incident to  $V(C)$  and
- (ii) there is at most one out-edge of  $v$  incident to  $V(C)$ .

Let  $D'$  be the digraph obtained by contracting the cycle  $C$  into one vertex  $c$ . Namely,  $D'$  has vertex set  $V(D') = (V(D) \cup \{c\}) \setminus V(C)$  and  $E(D')$  is the union of the following sets:

- $E(D[V(D) \setminus V(C)])$ ,
- $\{\overrightarrow{xc} : \exists \overrightarrow{xy} \in E(D), x \notin V(C), y \in V(C)\}$  and
- $\{\overrightarrow{cx} : \exists \overrightarrow{yx} \in E(D), x \notin V(C), y \in V(C)\}$ .

Note that properties (i) and (ii) imply that  $e(D') = e(D) - e(C) = e(D) - |V(C)|$ .

Suppose  $D'$  contains a transitive cycle  $\overrightarrow{TC}_k$  on vertices  $v_1, \dots, v_k$  for some  $k \geq 3$ . Namely,  $\overrightarrow{v_j v_{j+1}} \in E(D')$  for every  $j \in [k-1]$  and  $\overrightarrow{v_1 v_k} \in E(D')$ . If  $c \neq v_i$  for every  $i \in [k]$  then  $v_1, \dots, v_k$  form a transitive cycle in  $D$ , a contradiction. Therefore,  $c = v_i$  for some  $i \in [k]$ .

For brevity we only consider the case  $i \neq 1, k$  (the cases  $i = 1$  and  $i = k$  can be handled with a similar argument). By the definition of  $D'$ , there exist  $c_1, c_2 \in V(C)$  such that  $\overrightarrow{v_{i-1}c_1}, \overrightarrow{c_2v_{i+1}} \in E(D)$ . Furthermore, there exists an oriented path  $u_1 \dots u_\ell$  from  $u_1$  to  $u_\ell$  such that  $u_1 = c_1, u_\ell = c_2$  and  $u_j \in E(C)$  for all  $j \in [\ell]$ . Then  $v_1 \dots v_{i-1}u_1 \dots u_\ell v_{i+1} \dots v_k$  is an oriented path from  $v_1$  to  $v_k$  in  $D$ . Together with  $\overrightarrow{v_1v_k} \in E(D)$ , this forms a transitive cycle in  $D$ , a contradiction.

Therefore,  $D'$  is  $\overrightarrow{TC}_k$ -free for all  $k \geq 3$ ; so by the induction hypothesis we have

$$e(D') \leq \max \left\{ 2(n - |V(C)|), \left\lfloor \frac{(n - |V(C)| + 1)^2}{4} \right\rfloor \right\}.$$

The inequalities  $n \geq 4$  and  $2 \leq |V(C)| \leq n$  imply

$$2(n - |V(C)|) + |V(C)| \leq 2(n - 1) \quad \text{and} \quad \left\lfloor \frac{(n - |V(C)| + 1)^2}{4} \right\rfloor + |V(C)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

In particular,  $e(D) = e(D') + |V(C)| \leq \max \left\{ 2(n - 1), \left\lfloor \frac{n^2}{4} \right\rfloor \right\}$ . This concludes the proof of the claim.  $\blacksquare$

Because of Claim 2.1 we may assume that  $D$  contains no double edges (which are induced oriented cycles on two vertices). Additionally, we may assume that the underlying graph of  $D$  contains no triangle, since such a triangle would either correspond to an induced oriented cycle or a transitive cycle  $\overrightarrow{TC}_3$  in  $D$ . By Mantel's theorem, a triangle-free graph on  $n$  vertices has at most  $\lfloor n^2/4 \rfloor$  edges, hence  $e(D) \leq \lfloor \frac{n^2}{4} \rfloor$ . This concludes the inductive step.  $\square$

## 2.2. Proof of Theorem 1.4.

We prove Theorem 1.4 using the following two lemmata.

**Lemma 2.2.** *Let  $\mathcal{F} \subseteq 2^{[n]}$ . If for every  $i \in [n]$  there are elements  $A, B \in \mathcal{F}$  such that  $A \setminus B = \{i\}$  then  $|\mathcal{F}| \geq \min\{2\sqrt{n}, n/2 + 1\}$ .*

Notice that Lemma 2.2 is not specifically about induced saturated families. We will discuss this further in Section 3.

**Lemma 2.3.** *Given a poset  $P$ , let  $\mathcal{F}_0 \subseteq 2^{[n_0]}$  be an induced  $P$ -saturated family. Suppose there is an  $i \in [n_0]$  such that there are no elements  $A, B \in \mathcal{F}_0$  satisfying  $A \setminus B = \{i\}$ . Then  $\text{sat}^*(n, P) \leq |\mathcal{F}_0|$  for every  $n \geq n_0$ .*

*Proof of Theorem 1.4.* Let  $P$  be a poset. Suppose that for every  $n \in \mathbb{N}$  and every induced  $P$ -saturated family  $\mathcal{F} \subseteq 2^{[n]}$  we have that for every  $i \in [n]$  there are elements  $A, B \in \mathcal{F}$  such that  $A \setminus B = \{i\}$ . Then Lemma 2.2 implies  $|\mathcal{F}| \geq \min\{2\sqrt{n}, n/2 + 1\}$ , and thus  $\text{sat}^*(n, P) \geq \min\{2\sqrt{n}, n/2 + 1\}$ .

Otherwise, there exists some  $n_0 \in \mathbb{N}$  and  $i \in [n_0]$  so that there is an induced  $P$ -saturated family  $\mathcal{F}_0 \subseteq 2^{[n_0]}$  such that there are no elements  $A, B \in \mathcal{F}_0$  with  $A \setminus B = \{i\}$ . Then Lemma 2.3 implies that  $\text{sat}^*(n, P) \leq K_P$  for every  $n \in \mathbb{N}$  where

$$K_P := \max\{|\mathcal{F}_0|, \text{sat}^*(m, P) : m < n_0\}.$$

$\square$

The rest of this subsection covers the proofs of Lemmata 2.2 and 2.3. For the proof of Lemma 2.2 we apply Theorem 1.7.

*Proof of Lemma 2.2.* Let  $D$  be a digraph with vertex set  $\mathcal{F}$  and edge set  $E(D)$  defined as follows: for every  $i \in [n]$ , choose precisely one pair  $A, B \in \mathcal{F}$  with  $A \setminus B = \{i\}$ ; add the edge  $\overrightarrow{AB}$  to  $E(D)$ . Thus  $D$  has exactly  $n$  edges. Note that the hypothesis of the lemma ensures  $D$  is well-defined.

**Claim 2.4.** *For every  $k \geq 3$ ,  $D$  is  $\overrightarrow{TC}_k$ -free.*

**Proof of the claim.** It suffices to show that given  $\{A_1, \dots, A_k\} \subseteq \mathcal{F}$  such that  $\overrightarrow{A_j A_{j+1}} \in E(D)$  for every  $j \in [k-1]$  then  $\overrightarrow{A_1 A_k} \notin E(D)$ . By definition of  $D$ , there are distinct  $i_1, \dots, i_{k-1} \in [n]$  such that

$$(1) \quad A_j \setminus A_{j+1} = \{i_j\} \quad \text{for every } j \in [k-1].$$

This implies  $A_j \subseteq A_{j+1} \cup \{i_j\}$  for every  $j \in [k-1]$ , and thus  $A_1 \subseteq A_k \cup \{i_1, \dots, i_{k-1}\}$ . Hence,

$$(2) \quad A_1 \setminus A_k \subseteq \{i_1, \dots, i_{k-1}\}.$$

Note that if  $\overrightarrow{A_1 A_k} \in E(D)$  then (2) implies that  $A_1 \setminus A_k = \{i_j\}$  for some  $j \in [k-1]$ . However, recall that for every  $i \in [n]$  there is exactly one edge  $\overrightarrow{AB}$  in  $D$  such that  $A \setminus B = \{i\}$ ; so (1) implies that  $\overrightarrow{A_1 A_k} \notin E(D)$ , as desired.  $\blacksquare$

By Claim 2.4, we can apply Theorem 1.7 to the digraph  $D$ . This yields

$$n = |E(D)| \leq \max \left\{ 2(|\mathcal{F}| - 1), \frac{|\mathcal{F}|^2}{4} \right\},$$

which implies  $|\mathcal{F}| \geq \min\{2\sqrt{n}, n/2 + 1\}$ .  $\square$

We now present the proof of Lemma 2.3.

*Proof of Lemma 2.3.* Observe that it is enough to prove that for every  $n \geq n_0$  there exists a family  $\mathcal{F} \subseteq 2^{[n]}$  such that

- (i)  $|\mathcal{F}| = |\mathcal{F}_0|$ ,
- (ii)  $\mathcal{F}$  is induced  $P$ -saturated and
- (iii) there are no elements  $A, B \in \mathcal{F}$  satisfying  $A \setminus B = \{i\}$ .

We proceed by induction on  $n$  and observe that the base case ( $n = n_0$ ) follows directly from the assumption in the statement of the lemma.

Given an induced  $P$ -saturated family  $\mathcal{F} \subseteq 2^{[n]}$  satisfying (i)-(iii) we consider the following function  $f$  from  $2^{[n]}$  to  $2^{[n+1]}$ :

$$f(A) := \begin{cases} A & \text{if } i \notin A \text{ and} \\ A \cup \{n+1\} & \text{if } i \in A. \end{cases}$$

We shall prove that the family  $\mathcal{F}' := f(\mathcal{F}) \subseteq 2^{[n+1]}$  satisfies (i)-(iii).

First, note that (i) follows directly since  $f$  is injective. Second, (iii) also follows easily, since every element  $f(A)$  either contains both  $i$  and  $n+1$  or neither of them. Actually, because of this last property one might say that  $n+1$  behaves as a ‘copy’ of  $i$  in  $f(2^{[n]})$ . We need to prove (ii), i.e., that  $\mathcal{F}'$  is induced  $P$ -saturated.

It is easy to check that  $f$  preserves the inclusion/incomparable relations between elements. More precisely, for every  $A, B \in 2^{[n]}$  we have

$$(3) \quad A \subseteq B \iff f(A) \subseteq f(B).$$

Therefore, if  $P'$  forms an induced copy of  $P$  in  $\mathcal{F}'$ , then  $f^{-1}(P') \subseteq \mathcal{F}$  forms an induced copy of  $P$  in  $\mathcal{F}$ . This means that, since  $\mathcal{F}$  is induced  $P$ -free,  $\mathcal{F}'$  must be induced  $P$ -free as well. It is left to prove that for any  $S \in 2^{[n+1]} \setminus \mathcal{F}'$ , there is an induced copy of  $P$  in  $\mathcal{F}' \cup \{S\}$ . There are four cases to consider depending on  $S$ . The first two cases are short and the fourth case is an easy consequence of the third. Let  $S \in 2^{[n+1]} \setminus \mathcal{F}'$ .

*First case:*  $i, n+1 \notin S$ . Note that  $S \subseteq 2^{[n]}$  and  $f(S) = S$ , thus  $S \notin \mathcal{F}$  (as otherwise we would have  $S \in \mathcal{F}'$ ). As  $\mathcal{F}$  is induced  $P$ -saturated, there exists an induced copy  $P'$  of  $P$  in  $\mathcal{F} \cup \{S\}$ . By (3) the set  $\{f(F) : F \in P'\}$  is an induced copy of  $P$  in  $\mathcal{F}' \cup \{S\}$ .

*Second case:*  $i, n+1 \in S$ . Set  $S^* := S \setminus \{n+1\}$ . Note that  $S^* \subseteq 2^{[n]}$  and  $f(S^*) = S$ , thus  $S^* \notin \mathcal{F}$  (as otherwise we would have  $S \in \mathcal{F}'$ ). As  $\mathcal{F}$  is induced  $P$ -saturated, there exists an induced copy  $P'$  of  $P$  in  $\mathcal{F} \cup \{S^*\}$ . By (3) the set  $\{f(F) : F \in P'\}$  is an induced copy of  $P$  in  $\mathcal{F}' \cup S$ .

*Third case:*  $i \in S$  and  $n+1 \notin S$ . For this case we use the assumption that there are no elements  $A, B \in \mathcal{F}$  such that  $A \setminus B = \{i\}$ . Let  $S^* := S \setminus \{i\} \in 2^{[n]}$ . We use the sets  $S$  and  $S^*$  to find elements  $A, B \in \mathcal{F}$  that will contradict (iii) (for the family  $\mathcal{F}$ ).

**Claim 2.5.** *Either  $\mathcal{F}' \cup \{S\}$  contains an induced copy of  $P$  or there exists an element  $A \in \mathcal{F}$  such that  $A \subseteq S$  and  $i \in A$ .*

**Proof of the claim.** Assume that there is no  $A \in \mathcal{F}$  satisfying the properties of the claim; notice that this implies  $S \notin \mathcal{F}$ . We need to prove that  $\mathcal{F}' \cup \{S\}$  contains an induced copy of  $P$ .

Since  $\mathcal{F}$  is induced  $P$ -saturated, there exists an induced copy  $P'$  of  $P$  in  $\mathcal{F} \cup \{S\}$  that contains  $S$ . We shall prove that  $\{f(F) : F \in P', F \neq S\} \cup \{S\}$  is an induced copy of  $P$  in  $\mathcal{F}' \cup \{S\}$ .

First, observe that (3) implies the set  $\{f(F) : F \in P', F \neq S\}$  is an induced copy of the poset  $P' \setminus \{S\}$ . If for every  $F \in P' \setminus \{S\}$  the relation (inclusion/incomparability) between  $F$  and  $S$  is the same as the one between  $f(F)$  and  $S$  then we are done. Thus it is enough to prove for every  $F \in P' \setminus \{S\}$  we have that

- (i) if  $S$  and  $F$  are incomparable then  $S$  and  $f(F)$  are incomparable,
- (ii) if  $S \subseteq F$  then  $S \subseteq f(F)$  and
- (iii) if  $S \supseteq F$  then  $S \supseteq f(F)$ .

Notice that (ii) follows directly from  $F \subseteq f(F)$ . It is easy to check that (i) also holds by recalling that  $i \in S$  and  $n+1 \notin S$ . So finally, for  $F \in P' \setminus \{S\}$  as in (iii), observe that  $i \notin F$  otherwise  $F$  would satisfy the properties of  $A$  in the statement of the claim. Then  $f(F) = F$  and (iii) holds. ■

The proof of the following claim is very similar. We include it for completeness.

**Claim 2.6.** *Either  $\mathcal{F}' \cup \{S\}$  contains an induced copy of  $P$  or there exists an element  $B \in \mathcal{F}$  such that  $S^* \subseteq B$  and  $i \notin B$ .*

**Proof of the claim.** Assume that there is no  $B \in \mathcal{F}$  satisfying the properties of the claim; notice that this implies  $S^* \notin \mathcal{F}$ . It remains to show that  $\mathcal{F}' \cup \{S\}$  contains an induced copy of  $P$ .

Since  $\mathcal{F}$  is induced  $P$ -saturated, there exists an induced copy  $P'$  of  $P$  in  $\mathcal{F} \cup \{S^*\}$  that contains  $S^*$ . We shall prove that  $\{f(F) : F \in P', F \neq S^*\} \cup \{S\}$  is an induced copy of  $P$  in  $\mathcal{F}' \cup \{S\}$ .

First, observe that (3) implies the set  $\{f(F) : F \in P', F \neq S^*\}$  is an induced copy of the poset  $P' \setminus \{S^*\}$ . If for every  $F \in P' \setminus \{S^*\}$  the relation (inclusion/incomparability) between  $F$  and  $S^*$  is the same as the one between  $f(F)$  and  $S$  then we are done. Thus, for every  $F \in P' \setminus \{S^*\}$  we must prove that

- (i) if  $S^*$  and  $F$  are incomparable then  $S$  and  $f(F)$  are incomparable,
- (ii) if  $S^* \supseteq F$  then  $S \supseteq f(F)$  and
- (iii) if  $S^* \subseteq F$  then  $S \subseteq f(F)$ .

For (ii), notice that  $S^* \supseteq F$  implies  $i \notin F$  and thus  $S \supseteq S^* \supseteq F = f(F)$ . It is easy to check that (i) also holds by recalling that  $i \in S$  and  $n+1 \notin S$ . Finally, for  $F \in P' \setminus \{S^*\}$  as in (iii), observe that  $i \in F$  otherwise  $F$  would satisfy the properties of  $B$  in the statement of the claim. Then  $f(F) = F \cup \{n+1\}$  and (iii) holds.  $\blacksquare$

To finish the proof of this case, suppose that  $\mathcal{F}' \cup \{S\}$  does not contain an induced copy of  $P$ . By Claims 2.5 and 2.6 there are elements  $A, B \in \mathcal{F}$  such that  $A \subseteq S$ ,  $i \in A$  and  $S \setminus \{i\} \subseteq B$ ,  $i \notin B$ . In particular, we have  $A \setminus B = \{i\}$ , a contradiction.

*Fourth case:*  $i \notin S$  and  $n+1 \in S$ . Let  $S' := (S \setminus \{n+1\}) \cup \{i\}$ . Recall that every element of  $\mathcal{F}'$  either contains both  $i$  and  $n+1$  or neither of them. This property ensures that

- (a)  $S' \notin \mathcal{F}'$ ;
- (b) for any  $A \in \mathcal{F}'$ ,  $A \subseteq S$  if and only if  $A \subseteq S'$ ;
- (c) for any  $A \in \mathcal{F}'$ ,  $S \subseteq A$  if and only if  $S' \subseteq A$ .

By the third case we have that  $\mathcal{F}' \cup \{S'\}$  contains an induced copy of  $P$ . Clearly (b) and (c) then imply that  $\mathcal{F}' \cup \{S\}$  contains an induced copy of  $P$ .  $\square$

### 2.3. Proof of Theorems 1.5 and 1.6.

Given a poset  $P$  with legs and an induced  $P$ -saturated family  $\mathcal{F} \subseteq 2^{[n]}$ , the following lemma already implies that  $\mathcal{F}$  has size at least  $n$ .

**Lemma 2.7.** *Let  $P$  be a poset with legs and  $\mathcal{F} \subseteq 2^{[n]}$  be an induced  $P$ -saturated family. Then, there is an injective function  $f: [n] \rightarrow \mathcal{F} \setminus \{\emptyset\}$  such that*

$$f(i) = \begin{cases} \{i\} & \text{if } \{i\} \in \mathcal{F} \text{ and} \\ H(i) & \text{if } \{i\} \notin \mathcal{F}, \text{ where } H(i) \text{ is the hip of an induced copy of } P \text{ in } \mathcal{F} \cup \{\{i\}\}. \end{cases}$$

*Proof.* Let  $\mathcal{F}$  be an induced  $P$ -saturated family. For  $\{i\} \notin \mathcal{F}$  we shall choose a suitable  $f(i) = H(i)$  in such a way that  $f$  is injective. Given  $i \in [n]$  with  $\{i\} \notin \mathcal{F}$ , let  $P'$  be an induced copy of  $P$  in  $\mathcal{F} \cup \{\{i\}\}$  that contains  $\{i\}$ . Observe that  $\{i\}$  can only be a leg of  $P'$ . Among all possible choices for  $P'$ , we pick it under the following conditions.

- (i) The leg  $L'$  of  $P'$  which is not  $\{i\}$  is taken so that  $|L'|$  is as large as possible;
- (ii) under (i), the hip  $H'$  of  $P'$  is taken so that  $|H'|$  is as small as possible.

Thus we define  $f(i) := H'$ . Note that since  $L'$  and  $\{i\}$  are incomparable,  $i \notin L'$ .

**Claim 2.8.**  $H' = L' \cup \{i\}$ .

**Proof of the claim.** Suppose for a contradiction that  $L' \cup \{i\} \subsetneq H'$ . If  $L' \cup \{i\} \in \mathcal{F}$  then the poset  $(P' \cup \{L' \cup \{i\}\}) \setminus \{H'\}$  is an induced copy of  $P$ , which contradicts (ii). Therefore  $L' \cup \{i\}$  is not in  $\mathcal{F}$  and so  $\mathcal{F} \cup \{L' \cup \{i\}\}$  contains an induced copy  $P''$  of  $P$  which uses the set  $L' \cup \{i\}$ .



Denote the legs of  $P''$  by  $L''_1, L''_2$ . If  $L' \cup \{i\}$  is not a leg in  $P''$  then  $L''_1, L''_2 \subset L' \cup \{i\}$ . This implies that  $(P' \cup \{L''_1, L''_2\}) \setminus \{\{i\}, L'\}$  is an induced copy of  $P$  in  $\mathcal{F}$ , a contradiction. Thus,  $L' \cup \{i\}$  is a leg, and we may assume  $L''_1 = L' \cup \{i\}$ .

Note that if  $L'$  and  $L''_2$  are incomparable, then  $(P'' \cup \{L'\}) \setminus \{L''_1\}$  would be an induced copy of  $P$  in  $\mathcal{F}$ , a contradiction. Thus,  $L'$  is comparable with  $L''_2$ . In particular  $L' \subset L''_2$ , otherwise we would have that  $L''_2 \subseteq L' \cup \{i\} = L''_1$ , which contradicts the fact that  $L''_1$  and  $L''_2$  are incomparable. Moreover, again because  $L''_1$  and  $L''_2$  are incomparable, we have that  $i \notin L''_2$ .

Finally, observe that  $(P'' \cup \{\{i\}\}) \setminus \{L''_1\}$  is an induced copy of  $P$  in  $\mathcal{F} \cup \{\{i\}\}$  with legs  $\{i\}$  and  $L''_2 \supset L'$ , which contradicts (i).  $\blacksquare$

Recall  $f(i) := H' = H(i)$ . We shall prove that  $f$  is injective. Suppose not, namely  $f(i) = f(j)$  for some  $i \neq j$ . If either  $\{i\} \in \mathcal{F}$  or  $\{j\} \in \mathcal{F}$  then we get a contradiction. Thus, we have  $\{i\}, \{j\} \notin \mathcal{F}$ , and hence there exist induced copies  $P' \subseteq \mathcal{F} \cup \{\{i\}\}$  and  $P'' \subseteq \mathcal{F} \cup \{\{j\}\}$  of  $P$  such that  $H$  is the hip of both and  $f(i) = f(j) = H$ . Say  $\{i\}$  and  $L'$  are the legs of  $P'$  while  $\{j\}$  and  $L''$  are the legs of  $P''$ . Because of Claim 2.8 we have that  $H = L' \cup \{i\} = L'' \cup \{j\}$ , and hence  $L'$  and  $L''$  are incomparable. This implies that  $(P' \cup \{L''\}) \setminus \{\{i\}\}$  is an induced copy of  $P$  in  $\mathcal{F}$ , a contradiction.  $\square$

Now we are ready to prove Theorem 1.5.

*Proof of Theorem 1.5.* Observe that for a poset  $P$  with legs, the inequality  $\text{sat}^*(n, P) \geq n+1$  follows directly by applying Lemma 2.7 and by noticing that the empty set is in every induced  $P$ -saturated family.

For  $P$  such that  $P$  and  $\bar{P}$  have legs, we proceed as follows. Let  $\mathcal{F} \subseteq 2^{[n]}$  be an induced  $P$ -saturated family and observe that  $\emptyset, [n] \in \mathcal{F}$ . The poset  $\bar{\mathcal{F}} \subseteq 2^{[n]}$  is induced  $\bar{P}$ -saturated and therefore we can apply Lemma 2.7 to both  $\mathcal{F}$  and  $\bar{\mathcal{F}}$  to obtain injective functions  $f$  and  $\bar{f}$  respectively. Since both  $P$  and  $\bar{P}$  have legs, it is easy to see that the hip of  $P$  is not a maximal element of  $P$ . It follows that  $f(i) \neq [n]$  for every  $i \in [n]$ . That is,  $f([n]) \subseteq \mathcal{F} \setminus \{\emptyset, [n]\}$ ; similarly  $\bar{f}([n]) \subseteq \bar{\mathcal{F}} \setminus \{\emptyset, [n]\}$ .

By taking  $[n] \setminus \bar{f}(i)$  for every  $i \in [n]$  we define a new injective function  $\tilde{f}$  from  $[n]$  to  $\mathcal{F} \setminus \{\emptyset, [n]\}$ :

$$\tilde{f}(i) := \begin{cases} [n] \setminus \{i\} & \text{if } \{i\} \in \bar{\mathcal{F}}, \text{ and} \\ H(i) & \text{if } \{i\} \notin \bar{\mathcal{F}}, \text{ where } [n] \setminus H(i) \text{ is the hip of an induced } \bar{P} \text{ in } \bar{\mathcal{F}} \cup \{\{i\}\}. \end{cases}$$

Now, since  $f$  and  $\tilde{f}$  are injective, proving that  $f([n]) \cap \tilde{f}([n]) = \emptyset$  yields that  $|\mathcal{F}| \geq 2n + 2$  by recalling that the sets  $\emptyset$  and  $[n]$  belong to  $\mathcal{F}$ .

Suppose  $f(i) = \tilde{f}(j)$  for  $i, j \in [n]$ . If  $\{i\} \in \mathcal{F}$  or  $[n] \setminus \{j\} \in \mathcal{F}$  then it is easy to check that such an identity is not possible. Hence, we have that  $f(i) = \tilde{f}(j) = H$  where:  $H$  is the hip of an induced copy  $P_1$  of  $P$  in  $\mathcal{F} \cup \{\{i\}\}$  and  $[n] \setminus H$  is the hip of an induced copy  $\bar{P}_2$  of  $\bar{P}$  in  $\bar{\mathcal{F}} \cup \{\{j\}\}$ . In particular, this means that there is an induced copy  $P_2$  of  $P$  in  $\mathcal{F} \cup \{[n] \setminus \{j\}\}$  where  $[n] \setminus \{j\}$  plays the role in  $P_2$  of one of the two maximal elements of  $P$ ;  $H$  is in  $P_2$  and dominates all elements of  $P_2$  except for the two maximal elements.

Let  $L^1$  be the leg of  $P_1$  other than  $\{i\}$ . Let  $L^2_1, L^2_2 \in \mathcal{F}$  be the legs of  $P_2$ ; in particular,  $L^2_1$  and  $L^2_2$  are incomparable and  $L^2_1, L^2_2 \subset H$ . This implies that for every  $A \in P_1 \setminus \{L^1, \{i\}\}$ ,  $L^2_1, L^2_2 \subset A$ . Thus,  $(P_1 \cup \{L^2_1, L^2_2\}) \setminus \{L^1, \{i\}\}$  is an induced copy of  $P$  in  $\mathcal{F}$ , a contradiction.  $\square$

Theorem 1.6 now follows by a simple application of Lemma 2.7 and Theorem 1.5, together with two upper bound constructions.

*Proof of Theorem 1.6.* Observe that  $\text{sat}^*(n, Y) = \text{sat}^*(n, \Lambda)$  where  $\Lambda := \bar{Y}$ . Let  $\mathcal{F} \subseteq 2^{[n]}$  be an induced  $\Lambda$ -saturated family. Since  $\Lambda$  has legs, Lemma 2.7 yields an injective function  $f$  from  $[n]$  to  $\mathcal{F} \setminus \{\emptyset\}$  with  $f(i) = \{i\}$  if  $\{i\} \in \mathcal{F}$  and  $f(i) = H$  otherwise, where  $H$  is the hip of an induced copy of  $\Lambda$  in  $\mathcal{F} \cup \{\{i\}\}$ . This already implies  $|\mathcal{F}| \geq n + 1$  as  $\emptyset \in \mathcal{F}$ .

Observe that  $f(i) \neq [n]$  for every  $i \in [n]$ , therefore, if  $[n] \in \mathcal{F}$  then  $|\mathcal{F}| \geq n + 2$ , as desired. Hence, assume that  $[n] \notin \mathcal{F}$ , which means that  $\mathcal{F} \cup \{[n]\}$  contains an induced copy of  $\Lambda$  that uses  $[n]$ . Let  $L_1, L_2$  be the legs of this copy of  $\Lambda$  and  $H$  be the hip. Assume there is an  $i \in [n]$  such that  $f(i) = H$ ; so  $H$  is the hip of an induced copy of  $\Lambda$  contained in  $\mathcal{F} \cup \{\{i\}\}$ . Let  $M$  be the maximal element of this copy of  $\Lambda$  and observe that  $\{L_1, L_2, H, M\}$  forms an induced copy of  $\Lambda$  in  $\mathcal{F}$ , a contradiction. Hence,  $f(i) \neq H$  for every  $i \in [n]$ , which means that  $H$  was not counted before and therefore  $|\mathcal{F}| \geq n + 2$ . This argument implies that  $\text{sat}^*(n, Y) \geq n + 2$ .

For the poset  $X$  observe that  $X$  and  $\bar{X} = X$  have legs, therefore Theorem 1.5 directly implies that  $\text{sat}^*(n, X) \geq 2n + 2$ . For the upper bounds, consider the posets

$$P := \{F \in 2^{[n]} : |F| \geq n - 1 \text{ or } F = \emptyset\} \quad \text{and} \quad Q := \{F \in 2^{[n]} : |F| \geq n - 1 \text{ or } |F| \leq 1\},$$

and notice that they are respectively induced  $Y$ -saturated and induced  $X$ -saturated. Furthermore,  $|P| = n + 2$  and  $|Q| = 2n + 2$ .  $\square$

We conclude this subsection by exhibiting a class of posets for which the bound in Theorem 1.5 is sharp up to an additive constant.

For any  $\ell \in \mathbb{N}$ , let  $\wedge_\ell$  denote the poset with the following properties:

- $\wedge_\ell$  has legs  $L_1, L_2$  and hip  $H_1$ ;
- $\wedge_\ell \setminus \{L_1, L_2\} = \{H_1, \dots, H_\ell\}$  where  $H_j \subset H_{j+1}$  for every  $j \in [\ell - 1]$ .

Similarly, for any  $\ell \in \mathbb{N}$ , let  $X_\ell$  denote the poset with the following properties:

- $X_\ell$  has legs  $L'_1, L'_2$  and  $X_\ell \setminus \{L'_1, L'_2\} = \vee_\ell$  where  $\vee_\ell := \bar{\wedge}_\ell$ .

Clearly  $X_1 = X$  and  $\wedge_2 = \Lambda$ . Moreover,  $\wedge_\ell, X_\ell$  and  $\bar{X}_\ell$  have legs for every  $\ell \in \mathbb{N}$ , and therefore Theorem 1.5 implies that  $\text{sat}^*(n, \wedge_\ell) \geq n + 1$  and  $\text{sat}^*(n, X_\ell) \geq 2n + 2$ . The next proposition states that these bounds are close to the exact values of  $\text{sat}^*(n, \wedge_\ell)$  and  $\text{sat}^*(n, X_\ell)$ .

**Proposition 2.9.** *For all integers  $n - 1 > \ell \geq 2$  we have,<sup>1</sup>*

$$\begin{aligned} n + 1 &\leq \text{sat}^*(n, \wedge_{\ell+1}) \leq n + 2^{\ell+1} - \ell - 1 \quad \text{and} \\ 2n + 2 &\leq \text{sat}^*(n, X_\ell) \leq 2n + 2^{\ell+1} - 2\ell. \end{aligned}$$

*Proof.* First, we consider  $\wedge_{\ell+1}$ . Let  $\mathcal{F} \subseteq 2^{[n]}$  be the family containing precisely the following sets:

- the empty set  $\emptyset$ ;
- $\{i\}$  for every  $i \in [n]$ ;
- all subsets of  $\{1, 2, \dots, \ell\}$ ;
- all proper supersets of  $[n] \setminus \{1, 2, \dots, \ell\}$ .

It is straightforward to check that  $\mathcal{F}$  has  $n + 2^{\ell+1} - \ell - 1$  elements and is induced  $\wedge_{\ell+1}$ -saturated. Similarly, the family  $\mathcal{F}' := \mathcal{F} \cup \bar{\mathcal{F}} \subseteq 2^{[n]}$  has  $2n + 2^{\ell+1} - 2\ell$  elements and is induced  $X_\ell$ -saturated.  $\square$

<sup>1</sup>Note that the case  $\ell = 1$  is covered by Theorem 1.6 since  $\wedge_2 = \Lambda$  and  $X_1 = X$ .

### 3. CONCLUDING REMARKS

The following example shows that the bound in Lemma 2.2 is essentially tight.

**Example 3.1.** Let  $n \in \mathbb{N}$  be a perfect square, i.e.,  $\sqrt{n} \in \mathbb{N}$ . Let  $A_s, B_t \subseteq [n]$  be defined as follows.

- $A_s := \{s\sqrt{n} + 1, s\sqrt{n} + 2, \dots, s\sqrt{n} + \sqrt{n}\}$  for every  $s \in [\sqrt{n} - 1] \cup \{0\}$ ;
- $B_t := [n] \setminus \{t, t + \sqrt{n}, t + 2\sqrt{n}, \dots, t + (\sqrt{n} - 1)\sqrt{n}\}$  for every  $t \in [\sqrt{n}]$ .

Let  $\mathcal{F}^* := \{A_s : s \in [\sqrt{n} - 1] \cup \{0\}\} \cup \{B_t : t \in [\sqrt{n}]\} \subseteq 2^{[n]}$ .

Observe that  $\mathcal{F}^*$  has  $2\sqrt{n}$  elements. Furthermore, for every  $i \in [n]$ , there exists exactly one pair of elements  $A, B \in \mathcal{F}^*$  such that  $A \setminus B = \{i\}$ . Namely, if  $i = s\sqrt{n} + t$  where  $s \in [\sqrt{n} - 1] \cup \{0\}$  and  $t \in [\sqrt{n}]$  then  $A_s \setminus B_t = \{i\}$ .

Note that the digraph  $D$  with  $V(D) = \mathcal{F}^*$  and edge set  $E(D) = \{\overrightarrow{AB} : A \setminus B = \{i\}, i \in [n]\}$  is precisely the balanced oriented bipartite graph (i.e., an extremal example for Theorem 1.7).

This example shows that if one can improve the lower bound in Theorem 1.4 by using the auxiliary digraph approach, then one will really need to use the fact that the digraph is generated by an induced  $P$ -saturated family (recall this was not assumed in the statement of Lemma 2.2). On the other hand, if Theorem 1.4 is close to being best possible, then Example 3.1 points in the direction of potential extremal examples. That is, is there some poset  $P$  such that there is a minimum induced  $P$ -saturated family  $\mathcal{F} \subseteq 2^{[n]}$  that is ‘close’ to the family  $\mathcal{F}^*$  in Example 3.1?

Another natural question is to characterise those posets  $P$  for which  $\text{sat}^*(n, P)$  is bounded by a constant. Observe that Lemma 2.3 provides a method for determining such posets. That is, if one can exhibit an  $n_0 \in \mathbb{N}$  and an induced  $P$ -saturated family  $\mathcal{F} \subseteq 2^{[n_0]}$  such that for some  $i \in [n_0]$  there are no elements  $A, B \in \mathcal{F}$  with  $A \setminus B = \{i\}$ , then  $\text{sat}^*(n, P) = O(1)$ .

Along the lines of this research direction, Keszegh, Lemons, Martin, Pálvölgyi and Patkós [12] conjectured the following. Given a poset  $P$  on  $[p]$ , let  $\dot{P}$  denote the poset on  $[p + 1]$  where  $\dot{P} := P \cup \{[p + 1]\}$  (i.e.,  $\dot{P}$  is obtained by adding an element to  $P$  which dominates all elements in  $P$ ).

**Conjecture 3.2.** [12]  $\text{sat}^*(n, P) = O(1)$  if and only if  $\text{sat}^*(n, \dot{P}) = O(1)$ .

Note that neither direction of Conjecture 3.2 has been verified except for some special cases (see [12, Theorem 3.6]). It would be interesting to investigate if Lemma 2.3 can help tackle Conjecture 3.2.

Finally, it is natural to consider induced saturation problems for families of posets. Given a family of posets  $\mathcal{P}$ , we say that  $\mathcal{F} \subseteq 2^{[n]}$  is *induced  $\mathcal{P}$ -saturated* if  $\mathcal{F}$  contains no induced copy of any poset  $P \in \mathcal{P}$  and for every  $S \in 2^{[n]} \setminus \mathcal{F}$  there exists an induced copy of some poset  $P \in \mathcal{P}$  in  $\mathcal{F} \cup \{S\}$ . We denote the size of the smallest such family by  $\text{sat}^*(n, \mathcal{P})$ . By following the proof of Theorem 1.4 precisely, one obtains the following result.

**Theorem 3.3.** For any family of posets  $\mathcal{P}$ , either there exists a constant  $K_{\mathcal{P}}$  with  $\text{sat}^*(n, \mathcal{P}) \leq K_{\mathcal{P}}$  or  $\text{sat}^*(n, \mathcal{P}) \geq \min\{2\sqrt{n}, n/2 + 1\}$ , for all  $n \in \mathbb{N}$ .

In light of Theorem 3.3 it is natural to ask whether an analogue of Conjecture 1.2 is true in this more general setting, or whether (for example) the lower bound on  $\text{sat}^*(n, \mathcal{P})$  in Theorem 3.3 is best possible up to a multiplicative constant.

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