A multipartite Hajnal-Szemerédi theorem

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Abstract. The celebrated Hajnal-Szemerédi theorem gives the precise minimum degree threshold that forces a graph to contain a perfect K_k-packing. Fischer's conjecture states that the analogous result holds for all multipartite graphs except for those formed by a single construction. Using recent results on perfect matchings in hypergraphs, we prove that (a generalisation of) this conjecture holds for any sufficiently large graph.

1 Introduction

The celebrated Hajnal-Szemerédi theorem [6] states that if k divides n then any graph G on n vertices with minimum degree $\delta(G) \geq (k-1)n/k$ contains a perfect K_k-packing[1]. This theorem generalised a result of Corradi and Hajnal [3], who established the case $k = 3$, and is best-possible in the sense that the theorem would not hold assuming any weaker minimum degree condition. More recently, a series of papers [1,2,8,9] determined the minimum degree thresholds which force a perfect H-packing in a graph for non-complete graphs H, culminating in the work of Kühn and Osthus [11], who essentially settled the problem by giving the best-possible such condition (up to an additive constant) for any graph H, in terms of the so-called *critical chromatic number*.

In many applications it is natural to instead consider packings in a multipartite setting, in which the analogous problem seems to be considerably more difficult. More precisely, let $V_1, \dots, V_k$ be pairwise-disjoint sets of n vertices each, and G be a k-partite graph with vertex classes $V_1, \dots, V_k$ (so G has vertex set $V_1 \cup \dots \cup V_k$ and each V_j is an inde-

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[1] A *perfect H-packing* in a graph G is a spanning collection of vertex-disjoint copies of H in G; other sources have referred to the same notion as a *perfect H-tiling* or *H-factor*.

pendent set in G). We define the *partite minimum degree* of G, denoted $\delta^*(G)$, to be the largest m such that every vertex has at least m neighbours in each part other than its own, so

$$\delta^*(G) := \min_{i\in[k]} \min_{v\in V_i} \min_{j\in[k]\setminus\{i\}} |N(v) \cap V_j|,$$

where $N(v)$ denotes the neighbourhood of v.

Fischer [5] conjectured that the natural multipartite analogue of the Hajnal-Szemerédi theorem should hold. That is, he conjectured that if $\delta^*(G) \geq (k-1)n/k$ then G must contain a perfect K_k-packing. This conjecture is straightforward for $k = 2$, as it is not hard to see that any maximal matching must be perfect. However, Magyar and Martin [13] constructed a counterexample for $k = 3$, and furthermore showed that their construction gives the only counterexample for large n. More precisely, they showed that if n is sufficiently large, G is a 3-partite graph with vertex classes each of size n and $\delta^*(G) \geq 2n/3$, then either G contains a perfect K_3-packing, or G is isomorphic to the graph $\Gamma_{n,3,3}$ defined in Construction 1 for some odd n which is divisible by 3.

The implicit conjecture behind this result (stated explicitly by Kühn and Osthus [10]) is that the only counterexamples to Fischer's original conjecture are the constructions given by the graphs $\Gamma_{n,k,k}$ defined in Construction 1 when n is odd and divisible by k. We refer to this as the modified Fischer conjecture. If k is even then n cannot be both odd and divisible by k, so the modified Fischer conjecture is the same as the original conjecture in this case. Martin and Szemerédi [15] proved that (the modified) Fischer's conjecture holds for $k = 4$. Another partial result was obtained by Csaba and Mydlarz [4], who gave a function $f(k)$ with $f(k) \to 0$ as $k \to \infty$ such that the conjecture holds for large n if one strengthens the degree assumption to $\delta^*(G) \geq (k-1)n/k + f(k)n$. However, for general k the validity of even an asymptotic version of Fischer's conjecture (*i.e.* assuming that $\delta^*(H) \geq (k-1)n/k + o(n)$) was unknown until recently, when the results described below were obtained.

2 New results

Keevash and Mycroft [7] used new results on perfect matchings in k-uniform hypergraphs[2] to deduce the following asymptotic result (which

[2] A *hypergraph* H consists of a vertex set V and an edge set E, where each edge $e \in E$ is a subset of V. The edges are not required to be the same size; if they are then we say that H is a *k-uniform hypergraph*, or *k-graph*, where k is the common size of the edges.

was also proved independently and simultaneously by Lo and Markström [12] using the ‘absorbing’ method.)

Theorem 2.1. *For any k and $\varepsilon > 0$ there exists n_0 such that any k-partite graph G whose vertex classes each have size $n \geq n_0$ with $\delta^*(G) \geq (k-1)n/k + \varepsilon n$ contains a perfect K_k-packing.*

An r-partite graph can only contain a K_k-packing for $r \geq k$, since otherwise we do not have even a single copy of K_k. Fischer’s conjecture pertains to the case $r = k$, but it is natural to ask also for an analogous result for the case $r > k$. By a careful analysis of the extremal cases of Theorem 2.1, we can prove an exact result answering both Fischer’s conjecture and also this more general question for large n. This is the following theorem, the case $r = k$ of which shows that (the modified) Fischer’s conjecture holds for any sufficiently large graph. (The graph $\Gamma_{n,r,k}$ referred to in the statement is defined in Construction 1.)

Theorem 2.2. *For any $r \geq k$ there exists n_0 such that for any $n \geq n_0$ with $k \mid rn$ the following statement holds. Let G be a r-partite graph whose vertex classes each have size n such that $\delta^*(G) \geq (k-1)n/k$. Then G contains a perfect K_k-packing, unless rn/k is odd, $k \mid n$, and $G \cong \Gamma_{n,r,k}$.*

We now give the generalised version of the construction of Magyar and Martin [13] showing Fischer’s original conjecture to be false.

Construction 1. *Suppose rn/k is odd and k divides n. Let V be a vertex set partitioned into parts $V_1, \ldots, V_r$ of size n. Partition each V_i, $i \in [r]$ into subparts V_i^j, $j \in [k]$ of size n/k. Define a graph $\Gamma_{n,r,k}$, where for each $i, i' \in [r]$ with $i \neq i'$ and $j \in [k]$, if $j \geq 3$ then any vertex in V_i^j is adjacent to all vertices in $V_{i'}^{j'}$ with $j' \in [k] \setminus \{j\}$, and if $j = 1$ or $j = 2$ then any vertex in V_i^j is adjacent to all vertices in $V_{i'}^{j'}$ with $j' \in [k] \setminus \{3 - j\}$.*

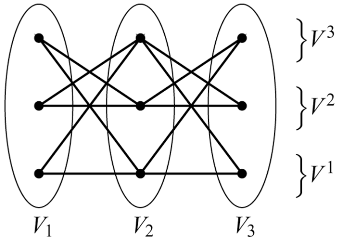

Figure 2.1. Construction 1 for the case $k = r = 3$.

Figure 2.1 shows Construction 1 for the case $k = r = 3$. To avoid complicating the diagram, edges between V_1 and V_3 are not shown: these are analogous to those between V_1 and V_2 and between V_2 and V_3. For $n = k$ this is the exact graph of the construction; for larger n we 'blow up' the graph above, replacing each vertex by a set of size n/k, and each edge by a complete bipartite graph between the corresponding sets. In general, it is helpful to picture the construction as an r by k grid, with columns corresponding to parts V_i, $i \in [r]$ and rows $V^j = \bigcup_{i\in[r]} V_i^j$, $j \in [k]$ corresponding to subparts of the same superscript. Vertices have neighbours in other rows and columns to their own, except in rows V^1 and V^2, where vertices have neighbours in other columns in their own row and other rows besides rows V^1 and V^2. Thus $\delta^*(G) = (k-1)n/k$. We claim that there is no perfect K_k-packing. For any K_k has at most one vertex in any V^j with $j \geq 3$, so at most $k-2$ vertices in $\bigcup_{j\geq 3} V^j$. Also $\left|\bigcup_{j\geq 3} \right| V^j = (k-2)rn/k$, and there are rn/k copies of K_k in a perfect packing. Thus each K_k must have $k-2$ vertices in $\bigcup_{j\geq 3} V^j$, and so 2 vertices in $V^1 \cup V^2$, which must either both lie in V^1 or both lie in V^2. However, $|V^1| = rn/k$ is odd, so V^1 cannot be perfectly covered by pairs. Thus G contains no perfect K_k-packing.

3 Rough outline of the proofs

As described above, Theorem 2.1, the asymptotic version of Fischer's conjecture, is proved by a short deduction from results on perfect matchings in uniform hypergraphs proved in [7]. Indeed, the result used gives fairly general conditions on a k-graph H which guarantee that either

(a) H contains a perfect matching, or
(b) H is close to a 'divisibility barrier', one of a family of lattice-based constructions which do not contain a perfect matching.

Given a graph G, we define the *clique k-complex* of G to be the hypergraph J on $V(G)$ whose edges are the cliques of size j in G for $1 \leq j \leq k$. Then a perfect K_k-packing in G is a perfect matching in the k-graph J_k consisting of all edges of J of size k. It is straightforward to show that if G meets the conditions of Theorem 2.1, then J_k satisfies the conditions necessary to apply the theorem from [7] described above. Furthermore, it is similarly not difficult to show that J_k is not close to a divisibility barrier, ruling out (b). So the theorem implies that (a) must hold, completing the proof of Theorem 2.1.

However, if we instead only assume that G satisfies the weaker conditions of Theorem 2.2, we can no longer deduce that J_k is not close to a divisibility barrier. Indeed, the clique k-complex of the graph $\Gamma_{n,r,k}$ constructed in Construction 1 is actually isomorphic to a divisibility barrier. On the other hand, if J_k is close to a divisibility barrier then we can obtain significant structural information regarding G. In fact, for $k \geq 3$ we find that we may partition G into two 'rows'. That is, we may find a subset U_i of each vertex class V_i of size pn/k for some $1 \leq p \leq k-1$ such that the bipartite graphs $G[U_i, V_j \setminus U_j]$ for $i \neq j$ are almost-complete. Except for a small number of 'bad' vertices, the rows $G_1 := G[\bigcup U_i]$ and $G_2 := G[\bigcup V_i \setminus U_i]$ satisfy a similar degree condition to G, but with p and $k-p$ respectively in place of k. This suggests our approach: we argue inductively to find a perfect K_p-packing in G_1 and a perfect K_{k-p}-packing in G_2. Using the fact that we have almost all edges between rows, we join each copy of K_p in the former packing to a copy of K_{k-p} in the latter packing to form a K_k-packing in G, as required.

However, for $k = 2$ there is another possibility for G for which J_k is close to a divisibility barrier. This is that G is *pair-complete*, meaning that we may choose $U_i \subseteq V_i$ of size $n/2$ for each i so that $G_1 := G[\bigcup U_i]$ and $G_2 := G[\bigcup V_i \setminus U_i]$ are almost-complete r-partite graphs, and there are very few edges in the bipartite graphs $G[U_i, V_j \setminus U_j]$. If there are in fact no edges in these bipartite graphs, and r and $n/2$ are both odd, then G cannot contain a perfect matching (*i.e.* perfect K_2-packing). This presents an obstacle to the proof strategy described above for $k \geq 3$ (since our inductive argument may fail for this reason). It transpires that we can avoid this problem by initially deleting a well-chosen small K_k-packing in G except for when G is exactly isomorphic to the graph $\Gamma_{n,r,k}$, and the theorem follows from this.

4 Future directions

As described in the introduction, the Hajnal-Szemerédi theorem on perfect K_k-packings in a graph G was followed by a sequence of papers addressing the problem of finding an H-packing in G for an arbitrary graph H. Following Theorem 2.2, it seems natural to ask for multipartite analogues of these theorems as well. In this direction, Martin and Skokan [14] recently proved an approximate multipartite version of the Alon-Yuster theorem. That is, they proved that if H is a graph with $\chi(H) \leq k$, and G is a k-partite graph with vertex classes $V_1, \dots, V_k$ of size n which satisfies $\delta^*(G) \geq (k-1)n/k + o(n)$, then G contains a perfect H-packing. One natural question is whether this minimum degree bound can be improved to include only a constant error term. Moreover, this bound is not even asymptotically best possible for many graphs: to find the degree threshold which forces a perfect H-packing in a k-partite graph for an arbitrary k-partite graph H an analogue of the critical chromatic number seems necessary.

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