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A proof of Sumner's universal tournament conjecture for large tournaments

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Abstract

Sumner's universal tournament conjecture from 1971 states that any tournament on $2n-2$ vertices contains any directed tree on n vertices. We prove that this conjecture holds for all sufficiently large n.

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1 Introduction

A tournament is an orientation of a complete graph. Obviously one cannot guarantee any substructures which contain a directed cycle within an arbitrary tournament. On the other hand, Sumner's universal tournament conjecture states that one can find any directed tree T within an arbitrary tournament G, even if the order of T is rather large compared to that of G. More precisely, the conjecture states that any tournament on $2n-2$ vertices contains any directed tree on n vertices. Many partial results towards this conjecture (made in 1971) have been proved – some of them are described below. In [11], we prove this conjecture for all large n.

Theorem 1.1 ([11]) *There exists n_0 such that the following holds. Let T be a directed tree on $n \geq n_0$ vertices, and G a tournament on $2n-2$ vertices. Then G contains a copy of T.*

To see that the bound is best possible, let T be a star with all edges directed inwards, and let G be a regular tournament on $2n-3$ vertices. Then every vertex of G has $n-2$ inneighbours and $n-2$ outneighbours, and so G does not contain a copy of T, whose central vertex has $n-1$ inneighbours. There are also 'near-extremal' examples which have a different structure to the one given above: let T be obtained from a directed path on $\ell \geq 1$ vertices by adding $y := (n-\ell)/2$ outneighbours to the terminal vertex of the path and y inneighbours to the initial vertex of the path. Let G consist of regular tournaments Y and Z, each on $2y-1$ vertices, together with an arbitrary tournament X on $\ell-1$ vertices so that all edges are oriented from Z to X, from X to Y and from Z to Y. Then $|G| = 2n-\ell-3$ as well as $|T| = n$, and it is easy to see that G does not contain T. These 'near-extremal' examples play a significant role in the proof of Theorem 1.1.

In [10], we used a randomised embedding algorithm to prove an approximate version of Sumner's universal tournament conjecture, and also a stronger result for directed trees of bounded degree. Both of these results are important tools in the proof of Theorem 1.1.

Theorem 1.2 ([10], Theorem 1.4) *Let $\alpha > 0$. Then the following properties hold.*

(i) There exists n_0 such that for any $n \geq n_0$, any tournament G on $2(1+\alpha)n$

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vertices contains any directed tree T on n vertices.

(ii) Let Δ be any positive integer. Then there exists n_0 such that for any $n \geq n_0$, any tournament G on $(1+\alpha)n$ vertices contains any directed tree T on n vertices with $\Delta(T) \leq \Delta$.

Let $f(n)$ denote the smallest integer such that any tournament on $f(n)$ vertices contains any directed tree on n vertices. So Sumner's conjecture states that $f(n) = 2n - 2$. Chung (see [17]) observed that $f(n) \leq n^{1+o(1)}$, and Wormald [17] improved this to $f(n) \leq O(n \log n)$. The first linear bound on $f(n)$ was established by Häggkvist and Thomason [4]. Havet [5] then showed that $f(n) \leq 38n/5$, and later Havet and Thomassé [7] used their notion of median orders to improve this to $f(n) \leq 7n/2$. Finally El Sahili used the same notion to prove the best known bound for general n, namely that $f(n) \leq 3n-3$. We make extensive use of this result (actually, any linear bound would suffice for our purposes; the factor of 3 is not essential.)

Sumner's conjecture is also known to hold for special classes of trees. In particular, Havet and Thomassé [7] proved it for 'outbranchings', again using median orders. Here an *outbranching* is a directed tree T in which we may choose a root vertex $t \in T$ so that for any vertex $t' \in T$, the path between t and t' in T is directed from t to t'. (Outbranchings are also known as arborescences.)

For many types of trees, Sumner's conjecture holds with room to spare. A classical result of this type is Redei's theorem.

Theorem 1.3 (Redei [14]) *Any tournament contains a spanning directed path.*

This was generalised considerably by Thomason [16] who showed that whenever n is sufficiently large, every tournament on n vertices contains every orientation of the path on n vertices (this was a conjecture of Rosenfeld). Havet and Thomassé [8] proved that this even holds for all $n \neq 3, 5, 7$. They also proposed the following generalisation of Sumner's conjecture :

Conjecture 1.4 (Havet and Thomassé, see [6]) *Let T be a directed tree on n vertices with k leaves. Then every tournament on $n + k - 1$ vertices contains a copy of T.*

Some special cases are known (see e.g. [2]). It would be interesting to know whether our methods can be used to prove this conjecture.

In the following section, we give a sketch of the proof of Theorem 1.2(ii). The proofs of Theorem 1.1 and 1.2(i) build on these ideas.

2 Sketch of proof of Theorem 1.2(ii)

The notion of a robust outexpander (which was introduced for dense graphs in [13]) is crucial to the proof. Informally, a digraph G is a robust outexpander if for any set $S \subseteq V(G)$ which is not too large or too small, the number of vertices with many inneighbours in S is substantially bigger than $|S|$. Kühn, Osthus and Treglown [13] showed that any robust outexpander G of linear minimum semidegree contains a Hamilton cycle. (Here the minimum semidegree is the minimum of the minimum indegree and the minimum outdegree.) Applying this to the 'reduced digraph' obtained from the Szemerédi regularity lemma implies that we can split most of the vertices of G into clusters $V_1, V_2, \dots, V_k$ so that the set of edges from V_i to V_{i+1} for each i (addition of the indices taken modulo k) forms a quasirandom and dense bipartite graph. As we shall see, this structure is very useful for embedding trees. On the other hand, it is easy to show that if a tournament G is not a robust outexpander of linear minimum semidegree, then the vertices of G can be split into two parts so that almost all of the edges between the two parts are directed the same way. We will then consider whether either of these two parts are robust outexpanders, and so on.

Then we show that Theorem 1.2(ii) holds with the added condition that G is a robust outexpander of linear minimum semidegree. So suppose the tournament G is a robust outexpander of linear minimum semidegree on $(1+\alpha)n$ vertices, and T is a directed tree on n vertices of bounded maximum degree. As described above, we can split most of the vertices of G into clusters $V_1, V_2, \dots, V_k$ so that the set of edges from V_i to V_{i+1} is quasirandom and dense for each i. Given this structure on G, one attempt to embed T in G would be to embed each vertex $t \in T$ in the cluster either preceding or succeeding the cluster containing the parent t' of t, according to the direction of the edge between t and t'. However, for many trees this method will fail to give an approximately uniform allocation of vertices of T to the clusters of G, which we require for the embedding to be successful. Instead, we modify this method so that each vertex is embedded as above with probability $1/2$ and is embedded in the same cluster as its parent with probability $1/2$. We show that with high probability this randomised algorithm will indeed give an approximately uniform allocation of vertices of T to the clusters of G, and so will successfully embed T in G.

It is a simple exercise to demonstrate that any transitive tournament on n vertices contains any directed tree on n vertices. We prove an analogue of this for almost-transitive tournaments G. This means that the vertices of

G can be ordered so that almost all of the edges of G are directed towards the endvertex which is greater in this order. We show that if G is an almost-transitive tournament on $(1+\alpha)n$ vertices and T is a directed tree on n vertices then G contains T.

Finally, we use the above results to prove Theorem 1.2(ii). So let G be a tournament on $(1 + \alpha)n$ vertices and let T be a directed tree on n vertices. If G is a robust outexpander of linear minimum semidegree, then our results show that G contains T, as desired. On the other hand, if G is not a robust outexpander of linear minimum semidegree then we may split G into two parts as described above (so almost all edges between these 2 parts are directed the same way). We now examine the larger of these two parts. If this is a robust outexpander of linear minimum semidegree then we stop; otherwise we again split this part into two. We continue in this fashion, always choosing the largest part of G, stopping if this is a robust outexpander and splitting it into two smaller parts if not. If we continue this process but do not find a robust outexpander of linear minimum semidegree, then G must be almost transitive. Indeed, each time we split G most of the edges across the split are directed the same way. So once all of the parts of G are sufficiently small, we can be sure that for some ordering of the vertices of G, almost all of the edges of G are directed according to this order. So G contains T, as desired.

So suppose instead that at some stage we stop because the largest part of G is a robust outexpander of linear minimum semidegree. Then we divide T into parts to be embedded amongst the parts of G, so that each part of G receives a part of T approximately proportional to its size. However, the robust outexpander part of G will actually receive slightly more vertices of T than it would from a proportional split. It turns out that our previous results guarantee that this part of T can still be embedded into the corresponding part of G. Since then the other parts of G will receive slightly fewer vertices of T than they would from a proportional split it will be possible to embed the remainder of T.

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