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Packing k-partite k-uniform hypergraphs

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Abstract

Let G and H be k-graphs (k-uniform hypergraphs); then a perfect H-packing in G is a collection of vertex-disjoint copies of H in G which together cover every vertex of G. For any fixed k-graph H let $\delta(H, n)$ be the minimum δ such that any k-graph G on n vertices with minimum codegree $\delta(G) \geq \delta$ contains a perfect H-packing. The problem of determining $\delta(H, n)$ has been widely studied for graphs (*i.e.* 2-graphs), but little is known for $k \geq 3$. Here we determine the asymptotic value of $\delta(H, n)$ for all complete k-partite k-graphs.

Keywords: hypergraphs, packings, codegree

1 Introduction.

A *k-uniform hypergraph*, or *k-graph* H consists of a *vertex set* $V(H)$ and an *edge set* $E(H)$, where every $e \in E(H)$ is a set of precisely k vertices of H. So a 2-graph is a simple graph. We write $|H|$ to denote the number of vertices of H. If G and H are k-graphs, then an *H-packing* in G (also known as an H-tiling or H-matching) is a collection of vertex-disjoint copies of H in G. This is a generalisation of a *matching* in G, which is the case of an H-packing

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when H has k vertices and one edge. A matching or H-packing in G is *perfect* if it covers every vertex of G.

We shall focus on the case when H is a fixed k-graph and $|G|$ is much larger than $|H|$. Our general question is then: what minimum degree is sufficient to guarantee that a k-graph G contains a perfect H-packing? There are several notions of minimum degree for k-graphs, but we shall consider here only one, which is often referred to as the *codegree*. So let G be a k-graph on n vertices. For any set $S \subseteq V(G)$ of size $k-1$, the *degree* $d(S)$ of S is the number of edges of G which contain S as a subset. The *minimum degree* $\delta(G)$ of G is then the minimum of $d(S)$ over all sets of $k-1$ vertices of G. Note that this coincides with the standard notion of degree for graphs. For any fixed k-graph H and any integer n we define $\delta(H,n)$ to be the smallest integer δ such that any k-graph G on n vertices with minimum degree $\delta(G) \geq \delta$ contains a perfect H-packing. Clearly this is only possible if $|H| \mid n$, and we only consider these values of n. Under this restriction, we wish to determine how $\delta(H,n)$ behaves for any fixed k-graph H when n is large.

For graphs this question has been widely studied, and the value of $\delta(H,n)$ has been determined up to an additive constant for any graph H. Indeed, the celebrated Hajnal-Szemerédi theorem [3] determined that $\delta(K_r,n) = (r-1)n/r$, and Komlós, Sárközy and Szemerédi [7] showed that for any graph H there exists a constant C such that $\delta(H,n) \leq (1-1/\chi(H))n + C$. This confirmed a conjecture of Alon and Yuster [1], who had proved this result with a linear error term. Finally Kühn and Osthus [8] determined the value of $\delta(H,n)$ up to an additive constant for any graph H by using the critical chromatic number $\chi_{cr}(H)$ first introduced by Komlós [6]. They showed that $\delta(H,n)$ depends on certain divisibility properties of H in a manner that is similar to our results for k-partite k-graphs.

On the other hand, for $k \geq 3$ far less is known. Until very recently even the asymptotic value of $\delta(H,n)$ was known only for the case of a perfect matching (*i.e.* a K^k_k-packing). The first bounds for this case were given by Daykin and Häggkvist [2]; Rödl, Ruciński and Szemerédi [11] later determined $\delta(K^k_k,n)$ precisely for all sufficiently large n (the value is always between $n/2-k$ and $n/2$). Very recently Keevash and Mycroft [5] used hypergraph matching results to show that $\delta(K^3_4,n) = 3n/4 + o(n)$. This proved an asymptotic version of a conjecture of Pikhurko [10], who had previously shown that $\delta(K^3_4,n) \leq 0.8603n$.

The results of the next section are taken from [9]. These provide upper bounds on $\delta(K,n)$ for any k-partite k-graph K. For complete k-partite k-graphs we provide matching lower bounds, thus determining the asymptotic value of $\delta(K, n)$ for any complete k-partite k-graph K.

2 Packing k-partite k-graphs.

Let K be a k-graph on vertex set U with at least one edge (if K has no edges then trivially $\delta(K, n) = 0$). A *k-partite realisation* of K is a partition of U into *vertex classes* $U_1, \dots, U_k$ so that for any $e \in K$ and $j \in [k]$ we have $|e \cap U_j| = 1$. Note in particular that we must have $|U_j| \geq 1$ for every $j \in [k]$. We say that K is *k-partite* if it admits a k-partite realisation. In a slight abuse of notation, we write *e.g.* "K is a k-partite k-graph on vertex set $U = U_1 \dot\cup \dots \dot\cup U_k$" to mean that K is a k-partite k-graph with vertex classes $U_1, U_2, \dots, U_k$. We say that a k-partite k-graph K on vertex set $U = U_1 \dot\cup \dots \dot\cup U_k$ is *complete* if every set $e \subseteq U$ with $|e \cap U_j| = 1$ for all $j \in [k]$ is an edge of K. Observe that a complete k-partite k-graph has only one k-partite realisation (up to permutations of the vertex classes $U_1, \dots, U_k$).

We categorise all k-partite k-graphs according to the divisibility relations between the sizes of their vertex classes. Indeed, let K be a k-partite k-graph on vertex set U. Then we define

$$\mathcal{S}(K) := \bigcup \{|U_1|, \dots, |U_k|\} \text{ and } \mathcal{D}(K) := \bigcup \{||U_i| - |U_j|| : i, j \in [k]\},$$

where in each case the union is taken over all k-partite realisations $U = U_1 \dot\cup \dots \dot\cup U_k$ of K. The *greatest common divisor* of K, denoted $\gcd(K)$, is then defined to be the greatest common divisor of the set $\mathcal{D}(K)$ (if $\mathcal{D}(K) = \{0\}$ then $\gcd(K)$ is undefined). We say that

(i) K is *type 0* if $\gcd(\mathcal{S}(K)) > 1$ or $|U_1| = |U_2| = \dots = |U_k| = 1$, and

(ii) for $d \geq 1$, K is *type d* if $\gcd(\mathcal{S}(K)) = 1$ and $\gcd(K) = d$.

Note that if $\gcd(K)$ is undefined then K is type 0. So every k-partite k-graph falls into one of these types. Our first theorem states that for any k-partite k-graph K, regardless of type, we have $\delta(K, n) \leq n/2 + o(n)$. Furthermore, Proposition 2.2 uses a well-known construction to show that this bound is asymptotically best possible for complete k-partite k-graphs of type 0.

Theorem 2.1 *Let K be a k-partite k-graph. Then for any $\alpha > 0$ there exists n_0 such that any k-graph G on $n \geq n_0$ vertices with $|K| \mid n$ and $\delta(G) \geq n/2 + \alpha n$ contains a perfect K-packing.*

Proposition 2.2 *Let K be a complete k-partite k-graph such that $p \mid \gcd(\mathcal{S}(K))$ for some $p > 1$. Then for any n there exists a k-graph G on n vertices with $\delta(G) \geq n/2 - k$ such that G does not contain a perfect K-packing.*

Proof. Let A and B be disjoint sets of vertices such that $|A|, |B| \geq n/2 - 1$, $p \nmid |A|$ and $|A \cup B| = n$, and let G be the k-graph on vertex set $A \cup B$ whose edges are all k-tuples $e \subseteq A \cup B$ such that $|e \cap A|$ is even. Then $\delta(G) \geq n/2 - k$. Also let $U = U_1 \dot\cup \ldots \dot\cup U_k$ be the (unique) k-partite realisation of K. Then since K is complete k-partite any copy of K in G must have either $U_j \subseteq A$ or $U_j \subseteq B$ for each $1 \leq j \leq k$. So in particular, the number of vertices in A covered by a given copy of K in G must be a multiple of p. Since $p \nmid |A|$ we may deduce that no K-packing in G covers every vertex of $|A|$. □

The next theorem shows that we have a much stronger bound on $\delta(K, n)$ for k-partite k-graphs of type 1. For any k-partite k-graph K, we define the *smallest class ratio* of K, denoted $\sigma(K)$, by

$$\sigma(K) := \frac{\min_{S \in \mathcal{S}(K)} S}{|V(K)|}.$$

So for any k-partite realisation of K, any vertex class of that realisation must have size at least $\sigma(K)|V(K)|$. Note that we must have $\sigma(K) \leq 1/k$. Theorem 2.3 shows that the parameter $\sigma(K)$ provides an upper bound on $\delta(K, n)$ for any k-partite k-graph K of type 1. If K is also complete k-partite, then Proposition 2.4 (another well-known construction) shows that $\delta(K, n)$ is actually asymptotically equal to this bound.

Theorem 2.3 *Let K be a k-partite k-graph with* $\gcd(K) = 1$. *Then for any $\alpha > 0$ there exists n_0 such that any k-graph G on $n \geq n_0$ vertices with $|K| \mid n$ and $\delta(G) \geq \sigma(K)n + \alpha n$ contains a perfect K-packing.*

Proposition 2.4 *Let K be a complete k-partite k-graph. Then for any n there exists a k-graph G on n vertices with $\delta(G) \geq \sigma(K)n - 1$ which does not contain a perfect K-packing.*

Proof. Let A and B be disjoint sets of vertices with $|A| = \sigma(K)n - 1$ and $|A \cup B| = n$, and let G be the k-graph on vertex set $A \cup B$ whose edges are all k-tuples $e \subseteq A \cup B$ with $|e \cap A| \geq 1$. Then $\delta(G) = |A| = \sigma(K)n - 1$. Also let $U = U_1 \dot\cup \ldots \dot\cup U_k$ be the k-partite realisation of K. Since K is complete k-partite, any copy of K in G must have $U_j \subseteq A$ for some $1 \leq j \leq k$. So every copy of K in G has at least $\min_{1 \leq j \leq k} |U_j| \leq \sigma(K)|K|$ vertices in A, and so any k-packing in G covers at most $|A|/\sigma(K) < n$ vertices of G. □

Finally, for k-partite k-graphs K of type $d \geq 2$ the next theorem show that $\delta(K, n)$ is bounded above by the larger of threshold of Theorem 2.3 and an additional threshold determined by $\gcd(K)$.

Theorem 2.5 *Let K be a k-partite k-graph with $\gcd(K) > 1$ and $\gcd(\mathcal{S}(K)) = 1$, and let p be the smallest prime factor of $\gcd(K)$. Then for any $\alpha > 0$ there exists n_0 such that any k-graph G on $n \geq n_0$ vertices contains a perfect K-packing if $|K| \mid n$ and*

$$\delta(G) \geq \max\{\sigma(K)n + \alpha n, \frac{n}{p} + \alpha n\}.$$

If K is complete k-partite, then Proposition 2.4 and the next proposition together show that the bound of Theorem 2.5 is asymptotically best possible. This is achieved through an interesting generalisation of the construction of Proposition 2.2. Indeed, the latter is the construction of Proposition 2.6 for $p = 2$.

Proposition 2.6 *Let K be a complete k-partite k-graph, and suppose that $p \geq 2$ satisfies $p \mid \gcd(K)$. Then for any n there exists a k-graph G on n vertices with $\delta(G) \geq n/p - k$ such that G does not contain a perfect K-packing.*

Proof. Let L_p be the $(p-1)$-dimensional sublattice of $\mathbb{Z}_p^p$ generated by the vectors $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$, where for each $1 \leq j \leq p-1$ we define

$$\mathbf{v}_j = (\overbrace{0, \dots, 0}^{j-1}, 1, \overbrace{0, \dots, 0}^{p-j-1}, j-1).$$

Then (†) for any vector $\mathbf{v} \in \mathbb{Z}_p^p$ there is precisely one coordinate of $\mathbf{v}$ which when incremented by one (modulo p) yields a vector $\mathbf{v}' \in L_p$. Take disjoint sets of vertices $V_1, \dots, V_p$ such that $V := \bigcup_{j=1}^p V_j$ has $|V| = n$. For any $S \subseteq V$ we define the *index vector* $\mathbf{i}(S)$ of S to be the vector in $\mathbb{Z}_p^p$ whose j-th coordinate is $|S \cap V_j| \bmod p$. Fix the sizes of $V_1, \dots, V_p$ such that $|V_j| \geq n/p - 1$ for each j and $\mathbf{i}(V) \notin L_p$, and let G be the k-graph on vertex set V whose edges are those k-tuples $e \subseteq V$ with $\mathbf{i}(e) \in L_p$. Then by (†) we have $\delta(G) \geq n/p - k$.

Since K is complete k-partite, by (†) any copy K' of K in G must have the property that $\mathbf{i}(e)$ is constant over all edges e of K'. Also, since $p \mid \gcd(K)$, and each coordinate of $\mathbf{i}(V(K'))$ is taken modulo p, it follows that $\mathbf{i}(V(K')) \in L_p$. So if a K-packing in G covers precisely $V' \subseteq V$, then we must have $\mathbf{i}(V') \in L_p$. Since $\mathbf{i}(V) \notin L_p$, there can be no perfect K-packing in G. □

The proofs of Theorems 2.1, 2.3 and 2.5 each use strong hypergraph regularity and the recent hypergraph blow-up lemma due to Keevash [4]. Indeed, the minimum degree conditions in each case are required to permit the deletion from G of a small number of copies of K to prepare G for the application of the blow-up lemma to complete the perfect K-packing.

We conclude by summarising our results for complete k-partite k-graphs K in tabular form. Let $p(d)$ denote the smallest prime factor of d. Then:

K is complete k-partite of	$\delta(K, n) =$
type 0	$n/2 + o(n)$
type 1	$\sigma(K)n + o(n)$
type $d > 2$	$\max\{\sigma(K)n, \frac{n}{p(d)}\} + o(n)$

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